Turnover and Dependency are Minimized when Population Growth is Negative

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Abstract

The sum of the birth rate and death rates is a measure of population turnover. It is also a measure of the care-giving needs – an alternative to the dependency ratio – since it tells the fraction of the population that is close to birth or close to death.

Here I show that the minimum turnover for stable populations is obtained when the population growth rate is slightly negative, exemplifying a case when negative population growth can be optimal.

An approximation of r^*, the population growth rate that minimizes turnover, shows that lower values of life expectancy and/or greater coefficients of variation of age at death make r^* more negative.

1 Introduction

Demographers are accustomed to considering the difference between crude birth and death rates, the growth rate of a population closed to migration. In this paper, I consider the sum of the birth and death rates, a measure showing how quickly the membership of the population is changing. This measure of turnover can also be used as a measure of dependency, since both newborns and those close to death are typically in need of large amounts of care.

The question considered here is what rate of population growth will minimize turnover. High population growth is the combination of high fertility and low mortality. Low population growth is the combination of low fertility and high mortality. To minimize the sum of birth and death rates, it turns out that a slightly negative population growth rate is optimal.

From a societal point of view, a social planner – taking the very narrow perspective of wanting to minimize the kind of dependency measured by the turnover rate – would aim for slightly sub-replacement fertility.

In this paper, I show some of the formal properties of the turnover of stable populations. These results apply some of the insights of stable population theory to the idea that the relevant metric of age is not just how far someone is from birth, but also how far they are from death. Early papers on "thanotological age" include Miller (2001). More recently, Sanderson and Scherbov (2010) have applied this idea to new measures of aging. Riffe (forthcoming) demonstrates ergodic results for populations defined by age from death. The distribution and properties of "life left" has also been the focus of a number of recent papers, such as Müller et al. (2004), Vaupel (2009), and Goldstein (2009).

This paper also relates to the work of Lee et al. (2014), who show that that optimal stable population growth rates vary by country and by the objective being optimized. Their work uses complete lifetime measures of economic profiles, rather than the approach here which uses only the beginning and the end of life.

2 Definitions and an illustration

Given a life table and a stable growth rate r, the turnover rate can be defined as the sum of the stable birth and death rates,

$$t(r) = b(r) + d(r). \tag{1}$$

The "turnover" aspect of t comes from the fact that b term represent the new arrivals and the d term represents the recent departures. If $b(r) = 3$ percent and $d(r) = 2$ percent, then 5 percent of the population changes from year to year and 95 percent of the population stays the same. The "care" or "dependency" interpretation of $t(r)$ comes from recognizing that those close to birth and/or close to death are typically in need of care. For the elderly, particularly as longevity increases, remaining life expectancy is a better than age for measuring health status and care needs. The death rate not only tells us what fraction of the population dies in a given year, it also tells us what fraction of the population is within a year of death.[1] Populations with large crude death rates also have a large demand for care of the sick and dying.

In stable populations there is a trade-off between birth and death rates. Consider the changes in stable growth driven by birth rates. In this case, higher growth rates lead to younger populations, which typically have lower death rates, and lower growth rates lead to older populations which typically have higher death rates.

Figure 1 illustrates the trade-off for the population with the life table of Swedish females for the period 1850-1859. We see, first, that as expected birth rates fall and death rates rise as r increases. Second, we see that the sum of birth and death rates is "U"-shaped, with extreme values of r leading to very high values of t, and an in-between value minimizing t. Finally, we see that the minimum value of t is not reached, as we might have expected, when $r = 0$, but rather when the growth rate is slightly negative, in this case about -2 percent.

3 Mathematical argument why turnover is minimized when population growth is negative

We prove that $t(r)$ is minimized when $r < 0$ by showing $t(r)$ falls with r for any $r \geq 0$.

Recall that

$$t(r) = b(r) + d(r).$$

By definition $r = b(r) - d(r)$, so we can rewrite turnover as

$$t(r) = 2b(r) - r.$$

Differentiating turnover with respect to r gives

$$t'(r) = 2b'(r) - 1 = 2A_{pop}(r)b(r) - 1$$

Consider the case when $r = 0$. In a stationary population, the birth rate is the inverse of the life expectancy at birth. So, $t(0) = 2A_{pop}/e_0 - 1$. It is

[1] Accounting for this year versus next year can be done by discounting by e^{-1r}. Thus $D(t)/N(t)$ is the crude death rate for year t. Whereas $D(t)/N(t-1) = D(t)/(N(t)e^{-1r}) = d(t)e^r$ is the fraction of last year's population dying this year, the fraction of the population within a year of dying.

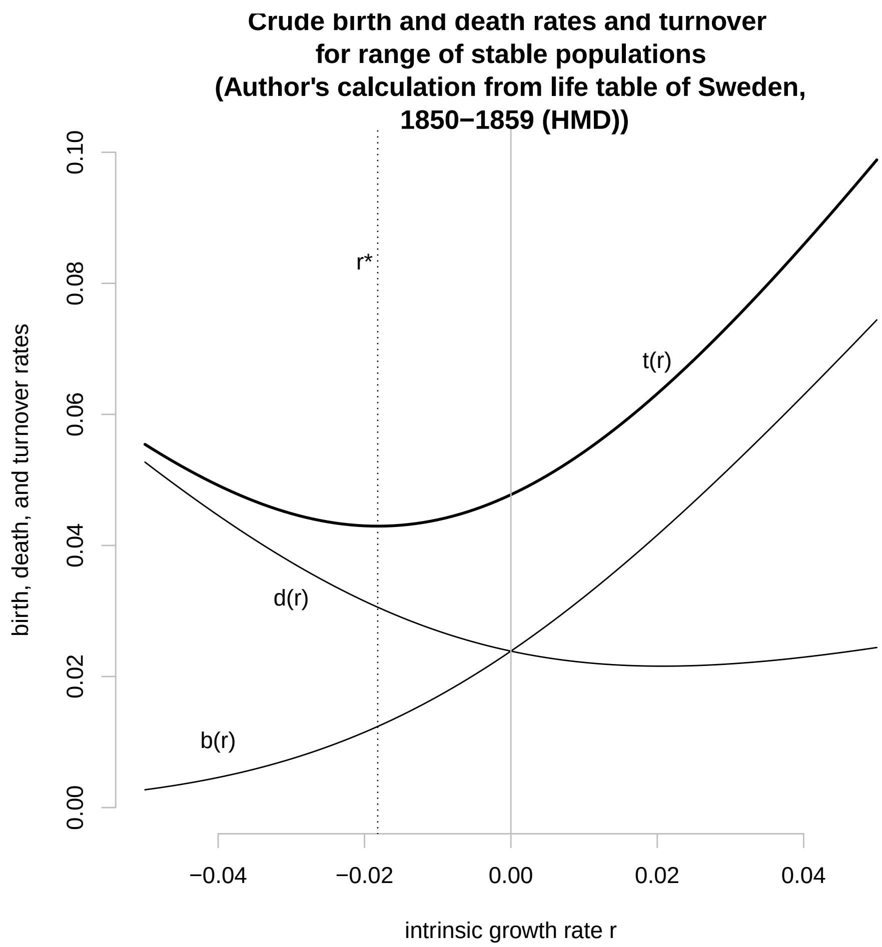


Figure 1: Example of how turnover depends on growth rate r

also the case that in stationary populations, the mean age of the population is greater than half of life expectancy as long as there some variance of age of death.[2] This means that $t'(0) > 0$, whenever there is variance in the age of death.

We can generalize this argument that $t' > 0$ to any $r > 0$ as follows. The derivative of the stable birth rate is for any r is

$$b'(r) = \frac{\int x\ell(x)e^{-rx}\,dx}{\left(\int \ell(x)e^{-rx}\,dx\right)^2}.$$

For positive r, we can define $\tilde{\ell}(x) = \ell(x)e^{-rx}$, with an accompanying density of deaths $\tilde{d}(x) = -\tilde{\ell}'(x)$, which is non-negative. Substituting in $\tilde{\ell}$, we can now re-apply the same argument we used above to all $r > 0$. As long as there is variance in the age at death, $t(r)$ will slope upward for all $r \geq 0$. This positive slope for $r \geq 0$ proves that $t(r)$ reaches its minimum when r is negative. [3] QED.

It is not necessarily the case that the turnover rate $t(r)$ will be U-shaped. For example, consider the case when the age-specific death rate is constant over all ages at hazard h. Here, any changes in r due to fertility will change the age-structure but this will not affect the population death rate, which will remain at h. Generally $b(r) = d(r) + r$. For constant hazards, this means $t(r) = 2h + r$, an upward-sloping straight line. Turnover will thus reach a minimum when r is as small as possible. In this case, r cannot be smaller than $-h$, since the smallest the birth rate can be is 0 and $r = b - d$. Thus, minimum turnover with constant hazards is reached when $\lim_{r \to -h} t(r) = h$. A similar monotonic decrease in $t(r)$ should also be found for any hazard function that decreases with age, since the aging of the stable population brought about by smaller r would not increase the population death rate. One could further explore the conditions on hazard functions needed to obtain U-shaped turnover with respect to r.

[2] The relationship is

$$\frac{A_{pop}(0)}{e_0} = \frac{1}{2}\left(1 + CV^2_{death}\right).$$

[3] I thank Ken Wachter for suggesting the extension of argument that $t'(r = 0) > 0$ to all positive r.

4 Minimizing turnover in human populations, an approximation and illustration

One can gain some analytic insight into what determines the growth rate that minimizes turnover by setting an approximation of the derivative of t to zero and obtaining an explicit, if approximate, expression for r^*.

Defining $G(r) = \frac{dt(r)}{dr}$, then for small absolute values of r, the first order Taylor expansion is

$$G(r) \approx G(0) + rG'(0).$$

Since r^* satisfies $G(r) = 0$, we have an approximation for the root as

$$\hat{r}^* = -\frac{G(0)}{G'(0)}.$$

This approach resembles the first step of Newton's method for iteratively finding the roots of an arbitrary function.

In this case, substituting appropriately, we have

$$\hat{r}^* = \frac{-CV_{death}^2}{2bA_{pop}^2(1 - CV_{pop}^2)} = \frac{-2CV_{death}^2}{e_0(1 + CV_{death}^2)^2(1 - CV_{pop}^2)}, \qquad (2)$$

where the last equality is obtained by substituting $1/e_0 = b(0)$, and using the relationship between A_{pop} and e_0.

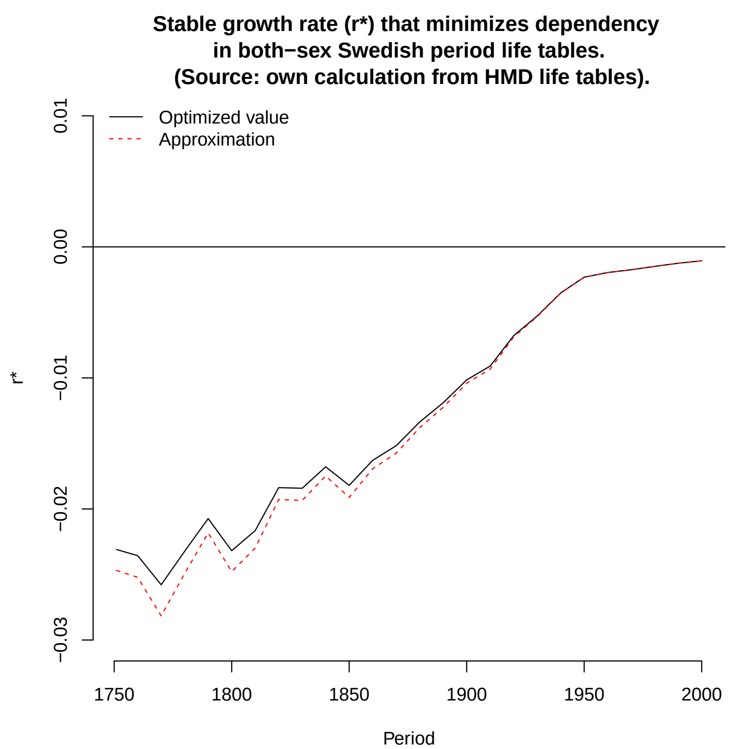
Inspecting the approximation, we see that it will be negative as long as the coefficient of variation of age of the population is less than unity, as is the case of most all human populations.

We can also gain some insight into how $\hat{r}^*$ depends on the life table. As life expectancy increases, the growth rate that minimizes turnover will move close to zero, since e_0 is in the denominator.

The relationship between the higher moments of the life table and turnover is less direct. However, empirically, the coefficient of variation of the age of the population is roughly constant, remaining at about 0.6 in Sweden from the mid-1700s to today. Taking CV_{pop} as constant, one can then see that $\hat{r}^*$ will become more negative the higher the coefficient of variation of age at death.

Over time, the optimum rate of population growth in terms of minimizing turnover, has remained negative but moved closer to zero. This has been due to both the fact that life expectancy has risen at the same time that the variance of age of death has fallen.

As an illustration, I have calculated the exact value of $r*$ and the approximate value for the available time series of Swedish life tables. We see that the approximation is very good, particularly when r^* is close to zero. The figure shows the numerically optimized values of r^* for the Swedish population by decade, as well as the approximated value using equation (2). The largest error occurs in the 1770s, when the optimization produces a value of -0.026 for r^*, whereas the approximation gives -0.028. For the most recent decades, the values are accurate to 5 decimal places.



The stable growth rate r^* that minimizes dependency and its approximation for Swedish both-sex decennial period life tables. Source: Own calculation from HMD life tables.

We can also ask what total fertility rate would correspond to the rate of intrinsic growth that minimizes turnover. Using the approximation

$$TFR^* = e^{\mu r^*}/(0.4886\ell_\mu),$$

where 0.4886 is the proportion female at birth, μ is the mean age of childbearing, and ℓ_μ is the probability of survival to the mean age of childbearing, the figure shows the optimal TFR for Sweden over the last centuries, letting $\mu = 30$, along with the replacement-level TFR in the corresponding stable population. We see that the optimal number of births from the point of view of minimizing turnover has remained remarkably constant over the centuries. The target growth rate has risen slightly but at the same time the fertility rate necessary to produce a given growth rate has fallen, with the net result being virtually no change in the total fertility rate needed to produce minimum turnover.

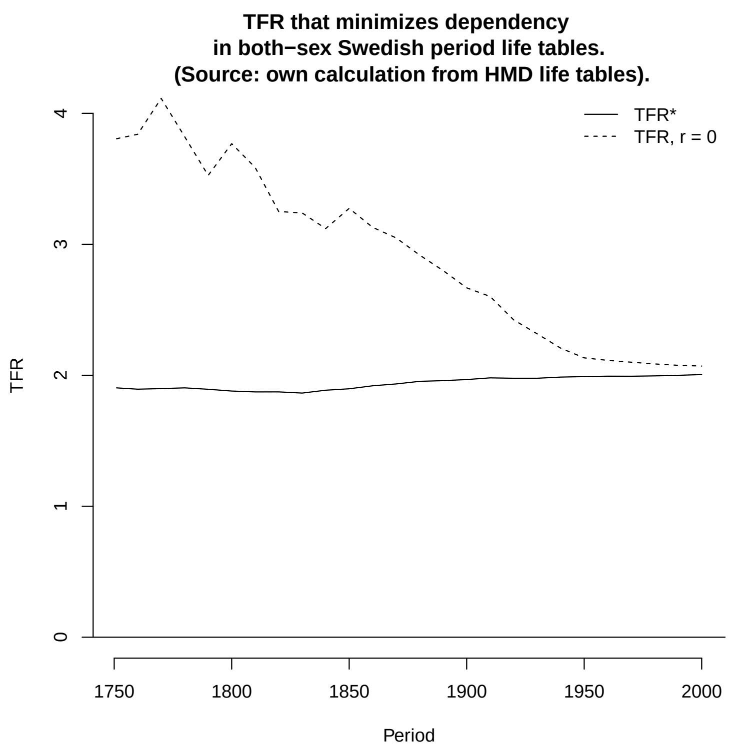


Figure 2: The total fertility rate that would minimize dependency in the stable population corresponding to the Swedish both-sex decennial period life tables and the associated replacement-level TFR. Source: Own calculation from HMD life tables.

Of course, one should not forget that this is for "stable" populations, and does not take into account the dynamics of changing population age structure over the course of the demographic transition. In actual demographic history, the period with the minimum dependency ratio would correspond some time after fertility rates had fallen but before the population had aged significantly, in the same manner as the traditional demographic dividend argument.

5 Extension to weighted dependency

In drawing a correspondence between turnover and dependency we have assumed that those close to birth and those close to death require the same amount of care. One can relax this assumption, by considering the weighted sum of birth and death rates, where the relative weights correspond to the relative care demands.

Define the weighted dependency rate as

$$t_w(r) = w_b b(r) + w_d d(r),$$

where the weights w_b and w_d provide the "costs" of births and deaths (or those near birth and death). These costs could be monetary, or time costs, or some other currency.

For convenience we can divide t_w by the sum of the weights. Letting, $w = w_b/(w_b + w_d)$, we have

$$t_w^*(r) = wb(r) + (1 - w)d(r).$$

Note that minimizing t_w^* will also minimize t_w.

Some algebra then shows us that the derivative of weighted dependency around stationarity is

$$t'^*_w|_{r=0} = \frac{1}{2}(1 + CV^2_{death}) - (1 - w),$$

which is minimized when $w = \frac{1}{2}(1 - CV^2_{death})$. For example, if $C = .5$, then the weighting $w = 3/8$ on births and $1 - w = 5/8$ on deaths, will make $r = 0$ the stable population with the minimum dependency rate. If we increase the "cost" of those near death relative to those just born, then it will be optimal to have fewer elderly, and thus the optimal growth rate will be higher than in the case where the weights are equal. Since we showed that $r* < 0$, when weights are equal, it makes sense that in order for the optimum growth rate to be zero, the weights on death would need to increase.

6 Conclusion

In this paper, I have defined a new quantity, the turnover rate, as the sum of the birth and death rates. Turnover is of interest not just as a measure of the speed of change of membership of a population but also because new arrivals (young children) and impending departures (the ill close to the death) require high amounts of care. Thus the sum of birth and death rates is a measure of dependency.

In stable populations, I showed that turnover is minimized when population growth is negative. A way to view this is to consider the population with zero growth and to ask whether the birth or death rates respond more to a marginal change in the growth rate. It turns out that the death rates are more responsive than the birth rates and so turnover can be minimized by reducing the death rate further, without a fully offsetting effect of increasing the birth rate.

Each life table has its own own rate of negative population growth that minimizes dependency. Over the range of human life tables, as illustrated by Swedish life tables from the mid-1700s to today, the rate of population growth that minimizes turnover has risen from about -3 percent to about -0.1 percent today. The increase in optimal population growth has been driven both by increasing life expectancy and by the lessening variance of age at death.

The ideas considered here could be extended in various ways. For example, rather than considering ages of birth and death, one could consider schedules of dependency, as Lee et al. (2014) do. Their argument could be extended to account for the relationship between dependency and years of remaining life. Another direction more closely linked to what we have done here would be to investigate further the effect of the variance of age at death on dependency and the accompanying change in optimal fertility. Another perspective that could be taken is that of the multigenerational family, in which the expected fraction of the family that is dependent will depend on demographic rates.