Stationary fertility trajectories and negative population momentum under continually falling mortality

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Abstract

Low-fertility countries face certain aging and possible population decline over the coming decades. There is strong interest in demographic policies that could slow or halt these changes, and concern about the negative momentum that is already built into national age structures. Mathematical models help us to understand the fertility changes that would be necessary to halt decline and minimize momentum effects, but standard analyses omit the effects of future mortality declines. We first study three distinct fertility trajectories that lead to stationarity in a closed population with constant mortality. We then analyze how these policies would play out if current mortality trends continued into the future. Surprisingly, we find that the same fertility paths can lead to stationarity when mortality falls continuously, although they do so over very long time spans and lead to unfamiliar, extremely old stationary age structures. We use both formal demographic methods and empirical simulations with German data. For an instantaneous transition to replacement fertility, plausible survival improvements could partially offset negative population momentum in age structure, but the population would still decline – for about the same number of years as in the classic, constant-mortality case. After that decline, a decreasing-mortality population would return to its initial population size, but only after 100 years or more. For less-familiar fertility trajectories leading to stationarity, we find similar results.

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1 Introduction

Fertility is too low. That was the recent view in 29 of 44 European countries (United Nations, 2014), 32 of which had policies intended to raise fertility level in 2013.[1] In Germany in 2010, for example, there were more than 140 federal and state regulations designed to promote fertility and family formation, with a total annual cost between 55 and 200 billion Euros (BMBFSFJ, 2013; prognos AG, 2014).

Population momentum adds an increased sense of urgency for policy makers in low-fertility societies. Recent history has already built age structures that work against population stabilization or rejuvenation. In any plausible medium-run scenario for future demographic rates, for example, Germany's population will likely get older and smaller.

Judgements about the desirability of fertility policies, alternative national population sizes, and age structures are subjective. Demographers cannot say what future is best for a population. We can, however, show what changes would be necessary to achieve specific objectives. We can also study the likely consequences of those policies in the short, medium, and long runs.

Using formal demographic analysis and simulations, we study three fertility trajectories that lead a closed population with constant age-specific mortality to stationarity (i.e., to a long-run asymptotic state with constant births, constant total population, and constant age structure). We focus on low-fertility populations that would require increased fertility levels to stabilize population levels, using Germany as an empirical example.[2]

After describing alternative paths to a stationary population, we then analyze how the same policies would play out if mortality continued to decline as expected by many experts. We demonstrate that the same fertility trajectories could lead to stationarity in a regime of ever-lower mortality and ever-longer lives. Via both mathematical analysis and simulation, we analyze the long-run properties and transient dynamics of populations that follow each of the stationary trajectories under plausible conditions of changing mortality. Relevant questions include

[1] Countries without pro-natalist policies, where the fertility level was considered "satisfactory", include very small states such as Andorra, Monaco, and San Marino.

[2] Immigration policy is another possible tool for halting population decline and stabilizing age structures. Schmertmann (2012) reviews the mathematical demography literature on immigration and stationarity, and United Nations Secretariat, Department of Economic and Social Affairs, Population Division (2001) provides empirical evidence on the volume of immigration that would be necessary to reach several related population targets. We do not consider immigration in this paper.

- What is the replacement level of TFR when mortality is falling continually?
- Under a stationary fertility policy, could increased old-age survival offset negative momentum? If so, how long would it take?
- How much lower are old-age support ratios on a stationary path when mortality falls?

Some of these questions address mathematical curiosities for demographers. Others are relevant for policy-makers considering alternative fertility policies against a backdrop of rapidly falling mortality and increased survival.

2 Stationary Fertility Trajectories with Constant Mortality

2.1 Notation

In order to analyze alternative demographic scenarios in a common framework, make the standard assumption of a closed, all-female population. Also define the following terms

$$
\begin{aligned}
N(a,t) &= \text{density of population at age } a \text{, time } t && (1)\\
B(t) &= \text{density of births at time } t && (2)\\
p_c(a,t) &= \frac{N(a,t)}{B(t-a)} = \text{cohort survival to age } a \text{ at time } t && (3)\\
\mu(a,t) &= -\left(\frac{\partial \ln p_c(a,t)}{\partial a} + \frac{\partial \ln p_c(a,t)}{\partial t}\right) = \text{mortality rate at age } a \text{, time } t && (4)\\
A_d(t) &= \frac{\int a\, p_c(a,t)\, \mu(a,t)\, da}{\int p_c(a,t)\, \mu(a,t)\, da} = \text{an average age of death at time } t && (5)\\
\theta(a) &= \text{standardized fertility at age } a \text{, scaled so that } \int \theta(\mathrm{a})\ \mathrm{da} = 1 && (6)\\
A_f &= \int a\, \theta(a)\, da = \text{period mean age of childbearing} && (7)\\
k(t) &= \text{fertility level at time } t && (8)\\
f(a,t) &= k(t)\, \theta(a) = \text{fertility rate at age } a \text{, time } t && (9)
\end{aligned}
$$

where (9) imposes a simplifying assumption that fertility can always be decomposed as the product of a standard shape function $\theta(a)$ and a scalar level $k(t)$. Unless otherwise noted, integration is always over ages $[0, \infty]$, with survival and fertility functions equal to zero above the maximum lifespan and age of menopause, respectively. If the sex ratio at birth is 105 sons per 100 daughters (assumed from here on), then two-sex period total fertility is $\mathrm{TFR}(t) = 2.05\, k(t)$; we will often report the fertility level using this more familiar measure.

Period births and deaths are

$$B(t) = \int N(a,t)\, f(a,t)\, da = \quad k(t) \cdot \int B(t-a)\, p_c(a,t)\, \theta(a)\, da \tag{10}$$

$$D(t) = \int N(a,t)\, \mu(a,t)\, da = \qquad \int B(t-a)\, p_c(a,t)\, \mu(a,t)\, da \tag{11}$$

Equation (10) relates the *birth trajectory* $\{B(t)\,,\, t > 0\}$ to the *fertility trajectory* $\{k(t)\,,\, t > 0\}$.[3] For the exposition that follows it is useful to note that if vital rates vary smoothly over ages, as they do in all relevant human populations, then (10) and (11) can be approximated as

$$B(t) \approx k(t)\, B(t - A_f)\, p_c(A_f, t) \tag{12}$$

$$D(t) \approx B(\, t - A_d(t)\,) \tag{13}$$

In the next subsections, we describe three alternative fertility and birth trajectories that would lead to stationary populations in a regime of constant age-specific mortality.

2.2 The Constant-Fertility Trajectory

Many classic papers in formal demography have analyzed how birth and fertility trajectories affect long-run population growth when mortality is constant. More than 250 years ago, Euler [1760, republished in English as Euler (1970)] showed that an exponentially changing birth sequence $B_t = ae^{rt}$ would lead, asymptotically, to exponential population growth at a fixed rate and to a fixed age structure. Stationarity (with $r = 0$ and constant births) is an important special case of Euler's results.

Lotka (1907; 1939) used continuous notation and empirically realistic patterns of age-specific vital rates to derive a much richer set of analytical results that included the feedback equation (10). An important corollary of Lotka's work is the following: there is only one unchanging level of fertility that leads to long-run stationarity. If mortality does not change, $p_c(a,t) = p_c^*(a)$ and births evolve according to a simplified version of (10):

$$B(t) = k(t) \int B(t-a)\, p_c^*(a)\, \theta(a)\, da \tag{14}$$

[3]For low positive values of t (i.e., early in the process under study), the right-hand sides of formulas (10) and (11) include births that occur before time $t = 0$. They can be inferred by rearranging (3) as $B(t-a) = \frac{N(0,a-t)}{p(a-t,0)}\ \forall a > t$.

The replacement level of fertility (i.e., the unique value that would lead the birth sequence to converge to a constant) in this case is [4]

$$k^* = \left[\int p_c^*(a)\, \theta(a)\, da \right]^{-1} \tag{15}$$

The integral in (15) can be well approximated by $p_c^*(A_f)$. Thus, in a contemporary, low-mortality population with nearly universal survival through menopause, $k^* \approx \frac{1}{p_c^*(A_f)}$ would be only slightly higher than one and the corresponding TFR* would be only slightly higher than 2.05.[5]

In a another classic paper, Keyfitz (1971) investigated how an instantaneous transition to k^* at time $t = 0$ would gradually lead to long-run stationarity, if fertility remained at that level forever afterwards. We will call this well-known scenario the *constant-fertility* path to stationarity. Keyfitz analyzed the transient dynamics of constant fertility, showing how the total population would evolve between the initial level at $t = 0$ and the final stationary level as $t \to \infty$. In particular, he demonstrated that the age structure of the initial population strongly affects the size of the long-run population. A population with a history of low fertility prior to a transition, such as contemporary Germany, will decline as it moves toward stationarity: for a while, deaths to the (relatively many) elders born long ago will exceed births to the (relatively few) young adults born recently, even if those young adults have replacement-level fertility. Population decline is thus programmed into a low-fertility population, in the form of *negative momentum*.

Figure 1 illustrates negative momentum and transitory dynamics under constant fertility, using demographic data from Germany in 2010. If Germany's fertility were set at $k^* = 1.008$ (TFR$^* = 2.07$) from 2010 onward, as in the Panel A, its female population would decline significantly, from an initial value of 41.7 million to a final stationary level of 34.1 million (Panel B). As the transition to stationarity progressed, births would fluctuate and stabilize at about 24% above their initial level (Panel C). The population's age structure would gradually approach the classical stationary form as annual births converged to a constant value.

[4]Long-run population growth is zero when $k(t) = k^*$ because only at that level are current births always a weighted average of past births, with non-negative age-specific weights $w(a)$ that integrate to one: $B(t) = \int B(t-a) \left[\frac{p(a)\theta(a)}{\int p(z)\theta(z)dz} \right] da = \int B(t-a)\, w(a)\, da$. See Arthur (1982) for an elegant, intuitive proof based on the constant averaging and re-averaging of births.

[5]Espenshade et al (2003) illustrate contemporary variations in replacement-level TFR due to mortality.

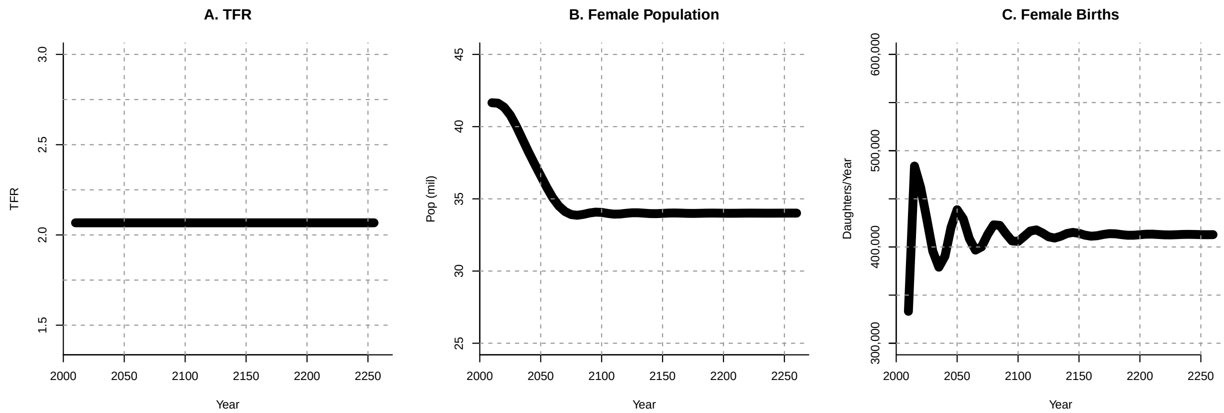

Figure 1: Constant-Fertility Trajectory for German Females at 2010 vital rates. Starting from the observed 2010 population, TFR equals replacement level of 2.07 (corresponding to k^*=1.008) from 2010 onward.

2.3 The Constant-Population Trajectory

Another option for reaching stationarity, discussed by Coale (1972) and Frejka (1972), would maintain total population constant after $t = 0$ by constantly adjusting fertility levels $k(t)$ to ensure that period births in (10) always equaled period deaths in (11). We call this the *constant-population* trajectory.

The constant-population scenario is, in a sense, the opposite of the Keyfitz momentum story: population is fixed from $t = 0$ onwards, but fertility levels oscillate in order to cancel out momentum in the age structure. Because deaths lag births by approximately one average lifetime, as in equation (13), the general pattern of the constant-population fertility trajectory is a damped wave with a period of about 80 years (cf. Coale 1972: Fig 3). Large, long (and slowly damped) oscillations in fertility would generate large changes in national age structure as long, alternating (and gradually smaller) baby booms and busts aged, died, and were exactly replaced by newborns.

Formally, one derives the constant-population fertility trajectory by assuming that mortality is unchanging $[p_c(a,t) = p_c^*(a),\, A_d(t) = A_d]$, equating (10) and (11) and solving for time-varying $k(t)$:

$$k^*(t) \;=\; \frac{\int\, B(t-a)\, p_c^*(a)\, \mu(a)\, da}{\int\, B(t-a)\, p_c^*(a)\, \theta(a)\, da} \;\approx\; \frac{B(t-A_d)}{B(t-A_f)\cdot p_c^*(A_f)} \tag{16}$$

Panel A of Figure 2 provides an empirical illustration of the constant-population fertility trajectory for Germany starting in 2010, with the corresponding population and birth trajectories in Panels B and C, respectively. It is intriguing to note that, even in a population like Germany's with an old initial age structure and considerable negative momentum, TFR would have to be significantly *lower* than 2.0 during much of a constant-population transition to stationarity. This occurs because canceling negative momentum initially requires high fertility and ever-larger birth cohorts (up to the year 2050 in the German case), but as these new cohorts pass through high-fertility, low-mortality ages 25–30 years later, fertility levels must fall to keep births equal to deaths. As a result, German TFR would have to be approximately 1.5 in the early 2070s in order to keep population constant. As the initial baby boom passes into high-mortality ages, fertility must again rise and the cycle repeats.

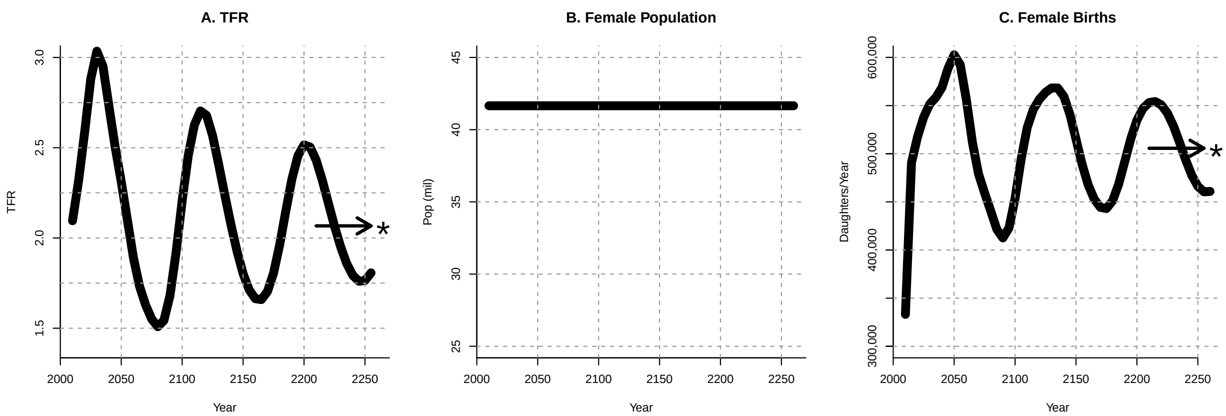

Figure 2: Constant-Population Trajectory for German Females at 2010 vital rates. Starting from the observed 2010 population, period fertility levels change in order to balance births and deaths at all times.

Convergence to stationarity is extremely slow on the constant-population trajectory. The plots in Figure 2 do not extend over long enough periods to illustrate convergence, but it is mathematically guaranteed by Preston's (1970) proof about necessary conditions for a long-run growth rate of zero under constant mortality. Preston showed that an asymptotic growth rate of zero for the total population (a condition that is trivially satisfied in the constant-population scenario) occurs only if the birth sequence converges to a constant.[6] From Preston's proof, we can infer births must eventually stabilize at $\bar{B} = \frac{\int \ N(0,a) \ da}{\int \ p_c^*(a) \ da}$, where the denominator is the unchanging life expectancy at birth, e_0. Because there is only one replacement level of fertility, $k^*(t)$ must therefore converge to a constant at the k^* value in

[6]Technically, Preston's proof requires non-zero fertility rates over a continuous interval or (in a model with discrete ages) positive fertility rates at integer ages that do not share a common divisor other than 1. These technical conditions are satisfied by all relevant human populations.

(15). In the German example, initial female population is 41.6 million, female e_0 is close to 83, and annual female births in Figure 2C will therefore reach an asymptotic limit near 500 thousand. These limiting values of TFR and B are indicated by asterisks at the right edges of Panels A and C, respectively.

2.4 The Constant-Birth Trajectory

A third path to stationarity, also discussed by Coale (1972) and Frejka (1972), is a fertility trajectory that leaves births at their $t = 0$ level for all $t > 0$. We call this a *constant-birth* trajectory, and the solution for this path under unchanging mortality is

$$k^*(t) = \frac{B(0)}{\int B(t-a)\, p_c^*(a)\, \theta(a)\, da} \approx \frac{B(0)}{B(t-A_f) \cdot p_c^*(A_f)} \tag{17}$$

With constant births from $t = 0$ onward, population size would continue to change for approximately 100 years due to momentum in the initial age structure. With constant births, age structure would converge much more smoothly than under the constant-population scenario, reaching the exact long-run structure after only one lifetime. Thus constant-birth fertility adjustment would lead to fast convergence in age structure, but at the cost of a smaller stationary population.

Figure 3 illustrates the fertility, total population, and birth trajectories associated with maintaining an unchanging number of births in Germany, starting as before in 2010. Panel A shows that the required path for TFR is a simple variant of the constant-fertility trajectory. Rather than an instantaneous switch to replacement (which would cause a sharp and immediate increase in the size of birth cohorts, as in Figure 1C), a constant-birth trajectory requires that the fertility level increase gradually from its initial level to the replacement level k^* over approximately A_f years. A_f=30.3 for Germany in 2010, and TFR(2010)=1.39, so the constant-birth path rises from 1.39 to the replacement level of 2.07 over a period of about thirty years. After that initial period of rising fertility, the cohorts passing through high-fertility ages will be equally large (by construction) and the fertility levels necessary to maintain constant births become constant at the long-run replacement level in (15).

In the constant-birth scenario, the gradual transition to replacement fertility occurs over approximately one reproductive lifetime. During this transitional phase in fertility, negative momentum in the initial population is only partially offset, and it is offset by less than in the constant-fertility trajectory. As a result, there is slightly more transitory aging and the final stationary population is smaller than in either of the other two trajectories.

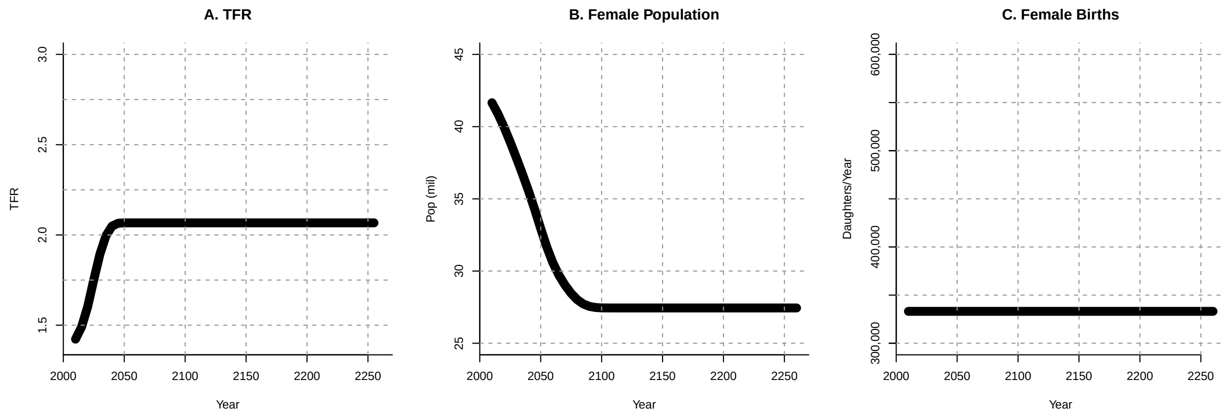

Figure 3: Constant-Birth Trajectory for German Females at 2010 vital rates. Starting from the observed 2010 population, fertility levels adjust to maintain a constant number of annual births.

On the other hand, the age structure converges much more quickly with the constant-birth trajectory than in the other two cases studied. Because birth cohorts are identical in size from $t = 0$ forward, a population on a constant-birth trajectory reaches stationarity within one lifespan, because after $t = \omega$ all living members of the population come from identically sized cohorts.

2.5 Summary

All three policies lead to long-run populations with constant births and the same age structure, but they differ radically in how they get there. Table 1 extends the graphical presentation with summary statistics for the three stationary policies under a regime of constant mortality. In the classic Keyfitz case of constant fertility and negative momentum (Constant Fertility) the German population would eventually shrink by more than 18 percent, despite a 24 percent increase in births. In contrast, if births were kept constant (Constant Births), then age structure would converge much more rapidly (in approximately 100 years, rather than 500), but the final stationary population size would be even smaller, at 34 percent lower than in 2010. Finally, the table shows that a constant-population fertility policy would halt population decline immediately, but that age structure would have to fluctuate significantly for thousands of years before converging to stationary form.

Table 1: Simulation results for stationary policies under constant mortality. Each column represents an alternative stationary fertility trajectory from Section 2. Total population is "steady" as soon as annual change is < 1 person. Age structure is "steady" as soon as annual change is < 1 person at every single year of age.

		Constant Fertility		Constant Population		Constant Births	
Total population steady after...		562 years		0 years		107 years	
Age distribution steady after...		532 years		2555 years		109 years	
Population Size (mil.)	2010	41.7		41.7		41.7	
	2050	37.0		41.7		33.4	
	2100	34.1		41.7		27.4	
	∞	34.1	(-18%)	41.7	(0%)	27.4	(-34%)
Annual Births (1000s)	2010	331		331		331	
	2050	442		606		331	
	2100	405		458		331	
	∞	412	(+24%)	504	(+52%)	331	(0%)
Support Ratio (SR=$\frac{P_{15-64}}{P_{65+}}$)	2010	2.76		2.76		2.76	
	2050	1.92		2.12		1.69	
	2100	2.57		2.65		2.59	
	Range after 2100	2.53–2.67		2.03–3.28		2.59–2.59	
	∞	2.59		2.59		2.59	

3 Stationary Trajectories with Declining Mortality

3.1 Asymptotic Stationarity

A population is stationary only if all age-specific growth rates are zero. Population density at age a and time t is $N(a,t) = B(t-a)\,p_c(a,t) = N(0,t-a)\,p_c(a,t)$, and its proportional growth rate (Arthur and Vaupel, 1984) is

$$\begin{aligned} g(a,t) &= \tfrac{\partial}{\partial t}\,[\ln\, N(a,t)] \\ &= \tfrac{\partial}{\partial t}\,[\ln\, N(0,t-a)] \;+\; \tfrac{\partial}{\partial t}\,[\ln\, p_c(a,t)] \\ &= g(0,t-a) \;+\; \tfrac{\partial}{\partial t}\,[\ln\, p_c(a,t)] \end{aligned} \tag{18}$$

Stationarity requires that $0 = 0 + \frac{\partial}{\partial t}\,[\ln p_c(a,t)]\;\forall\, a$, and therefore also requires constant cohort survival and mortality schedules.

However, stationarity is a limiting state, not a short-run condition. In the Keyfitz (1971) momentum model illustrated in Figure 1, for example, the population approaches stationarity as $t \to \infty$, rather than reaching it exactly at any finite time. Asymptotic stationarity implies that

$$\lim_{t\to\infty}\, g(a,t) \;=\; \lim_{t\to\infty}\, g(0,t-a) \;+\; \lim_{t\to\infty}\left[\tfrac{\partial}{\partial t}\ln\, p_c(a,t)\right] \;=\; 0 \qquad \forall\, a \tag{19}$$

which is possible in an environment of changing mortality, as long as change in cohort survival probabilities eventually approaches zero at all ages.

In the next subsection we describe a model of continuous mortality change, motivated by recent empirical research, for which $\lim_{t\to\infty}\left[\frac{\partial}{\partial t}\ln\, p_c(a,t)\right] = 0$ at all ages. In this model, all terms in (19) are zero, and the long-run population is stationary if annual births converge to a constant level.

In Section 3.2 we describe a model of continuous mortality change and calibrate it to German data. We then discuss the model's theoretical properties as $t \to \infty$ in Section 3.3. Finally, in Section 3.4 we explore the short-run dynamics (i.e., over first 250 years) of the three stationary fertility policies, in combination with two model mortality scenarios.

3.2 Mortality Model

Age-specific death rates may have been approximately constant for most of human history. However, life expectancy began to increase in the early 1800s, and has increased steadily in most developed countries since 1900 (Oeppen and Vaupel, 2002; Tuljapurkar et al, 2000;

White, 2002). Reductions in infant mortality caused most of the increase until the mid-twentieth century; in recent decades the primary factor has been increased survival at ages 65+ (Christensen et al, 2009). In our model we assume that this increase will continue, supported by several observations:

- If human life expectancy were close to biological limits, we would expect slower survival improvement in countries with low old-age death rates, and vice versa. No such positive relationship exists (Kannisto et al, 1994; Rau and Vaupel, 2014).
- Mortality reductions are reaching higher and higher ages (Rau et al, 2008).
- Studies of populations with especially healthy life styles indicate that life expectancy can still increase without any fundamental changes in the aging process (e.g., Andreev et al, 2011; Enstrom and Breslow, 2008; Winkler-Dworak, 2008).

We base our empirical illustration on data for German females in 2010. To avoid implausible trends we smooth the age-specific death/exposure ratios with two procedures: (1) at ages below 75, we use the P-spline smoothing approach of Eilers and Marx (1996) as implemented for mortality analysis by Camarda (2009; 2012); (2) at ages higher than 75 we use a parametric model $\mu(a) = \frac{\alpha e^{\beta a}}{\left(1+\frac{\alpha}{G}\left(e^{\beta a}-1\right)\right)}$.[7]

We use $\mu_0(a)$ to represent fitted mortality in 2010, and extrapolate period rates t years into the future as follows:

$$\mu(a,t) = \begin{cases} \mu_0(a) & \text{if } a \in [0,s] \\ \mu_0(s) & \text{if } a \in [s, s+\rho t] \\ \mu_0(a-\rho t) & \text{if } a \in [s+\rho t, \infty) \end{cases} \tag{20}$$

This assumes that there will be no future mortality improvement at ages below s, the age of onset of senescence in the original mortality schedule. We chose s=25 in our empirical

[7]This assumes Gompertz-distributed individual hazards with parameters α and β and a Gamma-distribution for frailty across individuals with a mean of one. This model was introduced by Vaupel et al (1979). Recent articles such as Steinsaltz and Wachter (2006) Missov and Finkelstein (2011), Missov and Vaupel (2015) support the notion that such a Gamma-Gompertz-Model (GGM) is not only attractive because of its relatively simple mathematical features, but is also reasonable for human mortality. The constant G is an upper limit for the mortality rate at extreme ages. We selected $G = 0.7$, the upper threshold of human mortality in non-parametric estimates by Gampe (2010).

We fit the parametric model for ages 75+ to German data via MLE, assuming Poisson distributed data (Brillinger, 1986).

example, because it is located at an almost-flat section of the mortality schedule – between the end of the "accident hump" of young adolescents and the onset of age-related mortality increase. We assume that all mortality improvement comes from delaying senescence, which is continually postponed by ρ years during every calendar year via a rightward shift of the age-specific mortality schedule.

Figure 4 illustrates our model with an example. The + symbols depict observed death/exposure ratios in the German 2010 data. The left solid line is the smoothed mortality curve, which serves as our baseline hazard $\mu_0(a)$.[8] The right solid line denotes mortality $t = 50$ years into the future at an annual postponement rate of $\rho = 0.2$ years, so that the cumulative shift is $\rho\, t{=}10$ years. Senescence still occurs at the later time, but age-specific mortality rates begin to rise at age 35 rather than 25. In colloquial terms: "35 is the new 25", "60 is the new 50", and so on.

We do not know how mortality will develop in the future, but three key aspects of our model reflect observed trends in recent decades. First, survival improvements have been reaching ever-higher ages – from infancy, to childhood, to adult ages (Christensen et al, 2009), and now even to nonagenarians (e.g., Rau et al (2008) for Japanese women). Second, we assume delayed, rather than decreased, senescence. The idea of such an age shift was introduced by Kannisto (1996) and further developed by Bongaarts and Feeney (2002, 2003), Bongaarts (2005, 2009), or Canudas-Romo (2008). Vaupel (2010, p. 538) corroborates the idea of shifting mortality, noting: "levels of mortality and other indices of health that used to prevail at age 70 now prevail at age 80, and levels that used to prevail at age 80 now prevail at age 90". The German Society of Actuaries also projects mortality using a baseline lifetable with an age shift (DAV, 2005). Finally, predicted limits to life expectancy have been a "trail of busted estimates" (Kennedy, 2004). Life expectancy increase has been happening for more than a hundred years (Oeppen and Vaupel, 2002).

Our model assumes that this regular trend will continue: the highest and lowest black lines in Figure 5 depict projected German life expectancy after 2010 in delayed senescence models with $\rho = 0.1$ or $\rho = 0.2$. They exhibit the expected linear increase. Figure 5 provides additional calibration information for ρ, by displaying historical trends in period life expectancy at birth for the UK and Germany through 2010, together with several existing forecasts. The trend in UK life expectancy is very similar to Germany's, and we include it because it provides a longer historical series. Forecasts using the delayed senescence model

[8] Although there is no obvious discontinuity at age 75, note that it is the breakpoint between the two separate smoothing approaches used to generate the fitted $\mu_0(a)$ schedule.

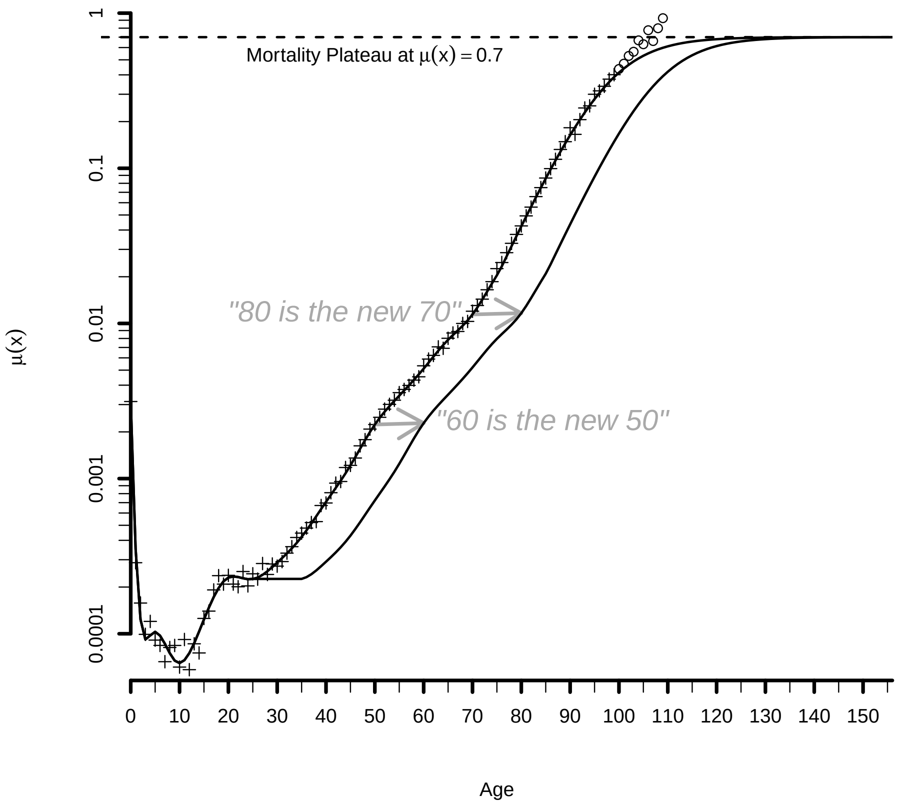

Figure 4: Fifty years of mortality change, in a delayed senescence model at annual postponement rate ρ=0.2. Mortality remains constant below age 25, and at higher ages the original schedule shifts rightward by 0.2×50=10 years. Observed death rates from 2010 (denoted by points) are smoothed as described in the text. Mortality rates at ages 100+ are denoted by small circles; population data at those ages were found to be unreliable in Germany (Scholz and Jdanov, 2006).

at ρ values of 0.1 and 0.2 bracket the 2060 forecasts for Germany produced by the Federal Statistical Office (H and L symbols), and also the trend forecasts from the UN and Eurostat.[9] Combining all of this information, we conclude that scenarios with ρ values of 0.10 and

[9] Bongaart's estimate (2009) for the period 1960–2000 for senescent mortality yields a very similar annual increase of 0.154 in senescent life expectancy (e_s). He notes that the "trend in e_s is close to linear between 1960 and 2000 and there is no obvious reason to believe that the pace will be significantly slower or faster in the future. This trend and the fact that e_0 converges on e_s in the long run makes senescent life expectancy the most suitable indicator for projecting future trends in life expectancy at birth" (Bongaarts, 2009, p. 212).

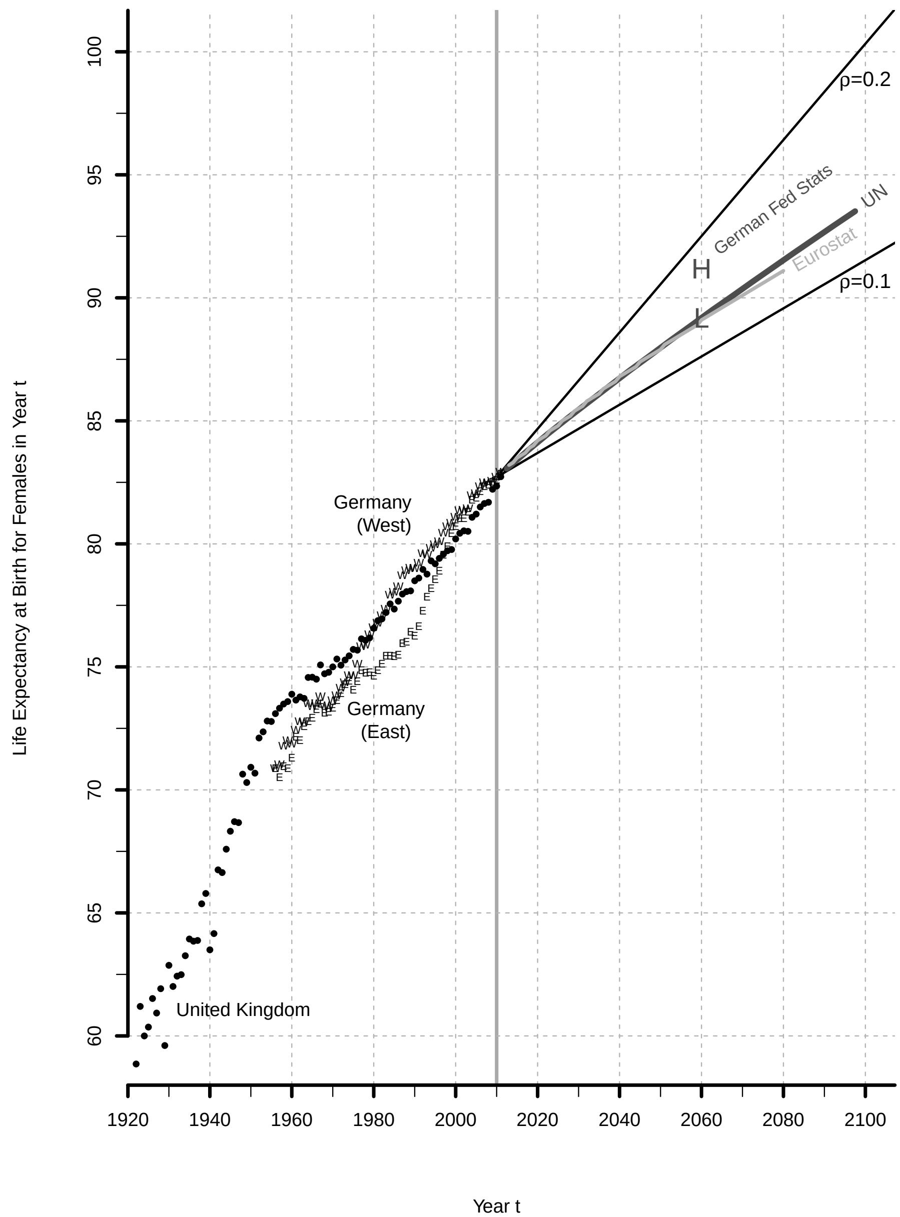

Figure 5: Observed life expectancy before 2010 in Germany (East & West separately) and the United Kingdom, plus projected life expectancy in the delayed senescence model at postponement rates ρ=0.1 and 0.2. Additional lines indicate future life expectancy as assumed by the United Nations and Eurostat. Points labeled H and L represent the German Federal Statistical Office's high and low projections, respectively, for female life expectancy at birth in 2060.

0.20 represent plausible lower and upper bounds, respectively, for delayed senescence in the empirical illustrations that follow.

3.3 Long-Run Analytics of Continually Declining Mortality

With continually delayed senescence, the effects of increased survival on age-specific growth in (18) eventually fall to zero at every age. At any point (a, t) on the mortality surface defined by Equation (20) the partial derivative with respect to time equals

$$\frac{\partial}{\partial t}\mu(a,t) = \begin{cases} -\rho\,\mu_0'(a-\rho t) & \text{if } t < \frac{a-s}{\rho} \\ 0 & \text{if } t \geq \frac{a-s}{\rho} \end{cases}$$

Thus at every age a there is a time $\frac{a-s}{\rho}$ after which mortality change ceases. Senescence implies that μ_0' is positive at ages above s, so mortality will decrease at age a until time $\frac{a-s}{\rho}$, and will remain constant afterwards. The limiting value of mortality depends on whether a is a pre-senescent age in the baseline schedule:

$$\lim_{t\to\infty} \mu(a,t) = \mu^*(a) = \begin{cases} \mu_0(a) & \text{if } a \leq s \\ \mu_0(s) & \text{if } a > s \end{cases} \tag{21}$$

With continual delays in senescence, there is an ever-expanding age range $[0, s+\rho\, t]$ over which both mortality rates and cohort survival probabilities become constant. Mortality rates in this age range reach the limits in (21), and survival probabilities therefore become locked in at $p_c^*(a) = \exp\left(-\int_0^a \mu^*(z)\,dz\right)$ for all cohorts reaching age a after time $\frac{a-s}{\rho}$. A population with constant births would therefore be stationary at ages below $s+\rho\, t$, and growing at higher ages. Below age $s+\rho\, t$ age-specific populations would be proportional to $p_c^*(a)$.

Before time $\frac{a-s}{\rho}$, as consecutive cohorts pass through age a, the later-born cohort will experience lower mortality, and therefore increasing cohort survival. After some mathematical manipulation, we find that age-specific growth rates in Equation (18) under continually delayed senescence are

$$g(a,t) = g(0, t-a) + \begin{cases} \left(\frac{\rho}{1-\rho}\right)\,[\mu_0(a-\rho t) - \mu_0(s)] & \text{if } t < \frac{a-s}{\rho} \\ 0 & \text{if } t \geq \frac{a-s}{\rho} \end{cases} \tag{22}$$

Understanding the model's implications for age-specific mortality change and growth provides insights into the (very) long-run dynamics of stationary fertility strategies under continually delayed senescence. Most importantly, in this model the positive effect of decreasing mortality on population growth eventually decreases to zero at *every* age. Thus, as $t \to \infty$ the age-specific growth rates in (22) all go to zero, and *in the limit* a constant number of annual births would produce a classical stationary population – even if senescence were delayed forever. This is identical to the usual stationary result with constant fertility and mortality schedules, which is also asymptotic as $t \to \infty$. However, in the case of continually delayed senescence the convergence process is much slower, and involves extremely long periods of gradually slowing growth among ever-older subpopulations.

Long-run stationarity is logically the same with or without delayed senescence, and also the same whether senescence delays are fast or slow. Thus we can use standard textbook results to deduce the limiting properties of populations under each of the three policies discussed earlier: in all cases, the total population will be $N = Be_0$, the crude birth and death rates will be $\frac{1}{e_0}$, and age specific populations will be proportional to (a very unfamiliar) $p_c^*(a)$. The three policies differ in their implications for scale of the limiting stationary population (N and B), but not for its structure.

Table 2 reports selected summary indices for the limiting populations as $t \to \infty$ when each of the three stationary policies is applied in a regime of continually delayed senescence. In all cases the long-run value of TFR is identical, at 2.05 $\left[\int p_c^*(a)\,\theta(a)\,da\right]^{-1} = 2.07$. Another notable feature is the extreme level of asymptotic e_0: if senescence were endlessly postponed, German female life expectancy at birth would eventually reach a limit of over 4000 years. This results from a life table based on the mortality rates in (21), and reflects the very high levels of survival that result as the phrase "*x is the new 25*" extends to ever-higher x values over time. High survival probabilities at extraordinarily high ages mean that the limiting stationary populations for all three policies would be extremely old. With falling mortality and an unchanging retirement age of 65, the stationary support ratio under all policies would eventually reach 1 worker for every 100 retirees (SR = 0.01), compared to 259 workers per 100 retirees (SR = 2.59) with mortality at 2010 levels.

The three policies lead to very different long-run population sizes. With a constant-fertility policy, TFR would decrease very slightly (due to falling mortality among prospective mothers) until senescence was postponed past the highest age of childbearing. Births would increase immediately, and approach an asymptotic level of 412 thousand per year almost as quickly as in Figure 1C. However, with increasing survival the elderly population would continue to grow for thousands of years. The end result would be a stationary female population of over 1.8 billion. A constant-population policy, by construction, eliminates population growth, but does so by reducing long-run births to extremely low levels: with continually delayed senescence a population of 42 million would require fewer than 10,000 births per year (less than 3 percent of Germany's 2010 level) to remain stationary. As before, the constant-birth population keeps annual births at a level below that reached under constant fertility, and the resulting asymptotic population is "only" 1.5 billion. Comparing the constant-fertility and constant-birth columns shows that a slower damping of the negative momentum in the 2010 population in the latter means approximately 350 million fewer German females in the long run.

Table 2: Demographic indices for German females in 2010, and their limiting values with alternative stationary policies under continually delayed senescence. Values in the last three columns are identical for any $\rho > 0$.

		Long-run limit with delayed senescence		
Measure	2010 Population	Constant Fertility	Constant Population	Constant Births
Female e_0 (years)	83	4,397	4,397	4,397
TFR	1.39	2.07	2.07	2.07
Total Female Population (mil)	42	1,811	42	1,458
Annual Female Births (1000s)	331	412	9	331
Crude Birth Rate (per 1000)	7.9	0.2	0.2	0.2
Crude Death Rate (per 1000)	10.8	0.2	0.2	0.2
Support Ratio $\frac{N_{15-64}}{N65+}$	2.76	0.01	0.01	0.01

3.4 Short-Run Results from Empirical Simulations

It is mathematically interesting to learn that populations can tend toward stationary even under conditions of declining mortality, but it is also important to understand what would happen under stationary policies on realistic time scales. In particular, we would like to know whether negative momentum in a low-fertility population could be offset by falling mortality, when it could be offset, and how much population aging would happen in the mean time. The next subsections examine these questions, for each of the three stationary policies in turn.

3.4.1 The Constant-Fertility Trajectory

Figure 6 illustrates the effects of mortality decline on a population following a constant-fertility trajectory.[10] Panel 6B shows how population momentum and growth are affected by increasing life expectancy. The lowest line in Figure 6B repeats the standard Keyfitz momentum calculation from Figure 1B, while the middle and upper lines show the population trajectory with mortality improvement at rates ρ=0.1 and 0.2, respectively.

Two forces operate on the total population size. Despite continuous survival improvements, negative momentum in the relatively old age structure initially produces an excess of deaths over births and drives population downward for approximately 60 years, regardless of survival changes. From $t = 0$ onward, increased survival at all ages is a positive influence on total population, but in all cases it is weaker than the initial momentum/age structure effects. After about 60 years, the negative force of momentum dies out (literally), the effects of increased survival begin to dominate, and the population begins to increase.

In the $\rho = 0.1$ case, the female population falls to 36.9 million, compared to 34 million in the constant-mortality scenario. If mortality is postponed by 0.2 years annually, the population size still declines, but only by about five percent, to 39.4 million people. In contrast to the standard population momentum scenario with constant mortality, population in the changing-mortality case does not approach an asymptotic limit, but begins increasing again after about sixty years. It would take about 100 years for total population to recover to its initial level if ρ=0.2, and almost twice as long if ρ=0.1.

The birth trajectories in both cases presented here (Panel C) are remarkably similar to the classic Keyfitz momentum case with constant mortality, presented earlier in Figure 1. The number of daughters required for a constant TFR fluctuates with a frequency of about 30 years, the average age at childbearing, and eventually converges to a level almost identical to that in Keyfitz model.

3.4.2 The Constant-Population Trajectory

Results from the constant-fertility projections demonstrate that negative momentum in the age structure cannot be completely offset by combining replacement-level fertility with plau-

[10] With declining mortality, the replacement fertility level implied by equation (15) falls slightly over time (from 2.0703 to 2.0697, converging faster when ρ is higher) because of small improvements in maternal survival at ages above $s = 25$. We include these changes in the "constant-fertility" scenarios, but they have no meaningful effects and are not visible in the figures.

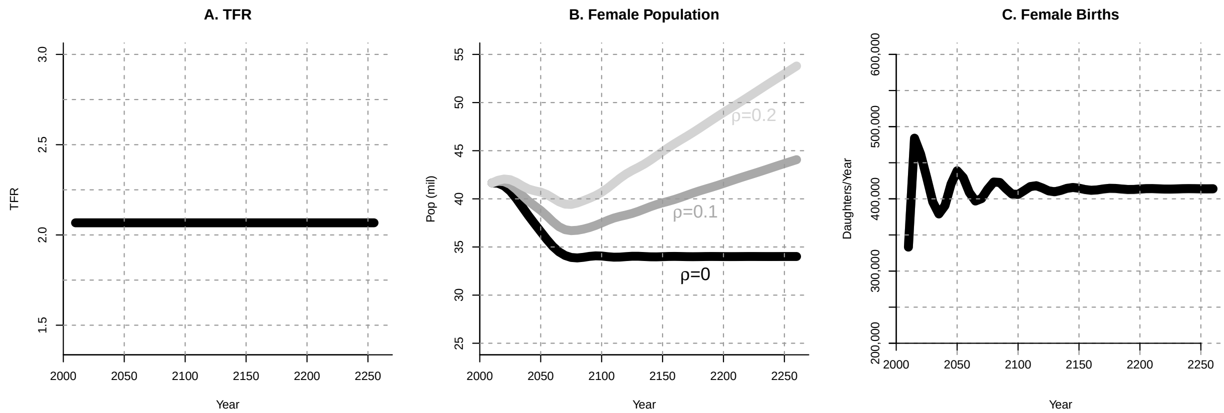

Figure 6: Constant-Fertility Trajectory, with alternative rates of delayed senescence. See Figure 1 caption for more details.

sible mortality improvements. Therefore, to achieve a constant population when mortality changes, fertility levels must still fluctuate in order to cancel momentum.

Recall that baby booms and busts on a constant-population trajectory would have a period approximately equal to life expectancy (Figure 2, Panels A and C). This occurs because an unchanging population requires that births must be high in periods with high numbers of deaths, and high numbers of deaths will tend to happen $A_d(t)$ years after a baby boom. Thus, we would expect cycles with a period close to $A_d(t)$, a measure similar to life expectancy. If mortality declines and average lives gradually get longer, then the period between baby booms should gradually lengthen if population is to stay constant. In other words, the cycles seen in Panels A and C of Figure 2 should be stretched horizontally, with ever-longer intervals between peaks as one moves farther into the future.

Our empirical findings for mortality postponement by $\rho = 0.1$ and by $\rho = 0.2$ years annually, in combination with a constant-population fertility policy, are presented in Figure 7. We include constant-mortality (ρ=0) curves for reference. As expected, delayed senescence and increased old-age survival lengthens the time between the peaks and troughs in the time series of fertility levels (Panel A). Figure 7 also shows that increased survival lowers the peaks in the TFR cycle: the more deaths are postponed, the lower the fertility level necessary to generate enough births to keep the population constant (as shown by the decreasing birth trajectories in Panel C). Asymptotic results in Table 2 show that female births in both of the $\rho > 0$ cases in Panel 7C will eventually stabilize at extremely low, off-the-chart levels — in fact, below 10 thousand per year.

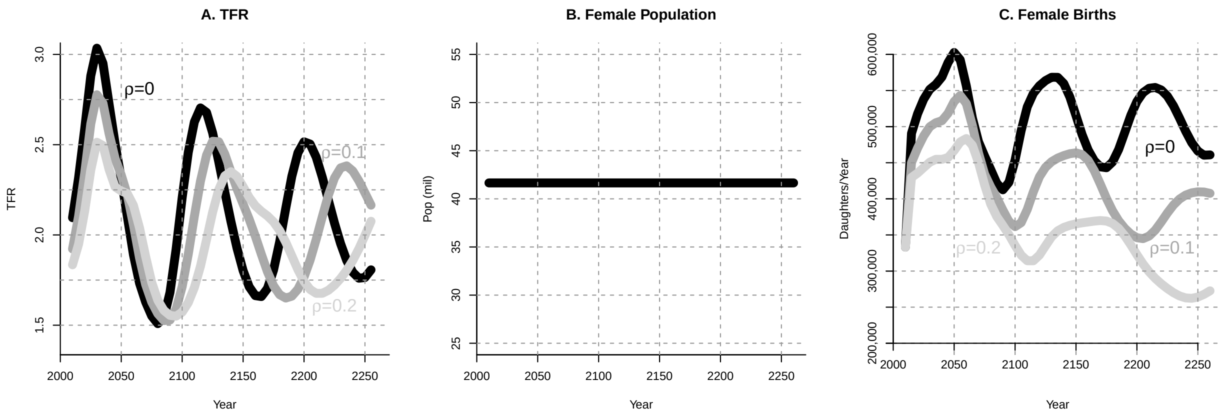

Figure 7: Constant-Population Trajectory, with alternative rates of delayed senescence and decreasing adult mortality. See Figure 2 caption for more details.

3.4.3 The Constant-Births Trajectory

When the policy objective is to maintain births at the initial level forever, rather than to have them continually offset period deaths, then the fertility trajectory is much smoother and simpler – even when mortality rates change. Figure 8 shows results for constant-birth fertility trajectories under various mortality scenarios. In all cases, fertility rates must rise smoothly to the replacement level over about 30 years. Reductions in maternal mortality have trivial effects on the constant-birth trajectory $k(t)$, but they are so small that they are invisible in Figure 8A, which contains three almost completely overlapping lines.

The population consequences of the constant-birth fertility path, shown in Figure 8B, are qualitatively similar to those in the constant-fertility case: population falls for approximately 60 years due to negative momentum, regardless of mortality changes, and then recovers and eventually increases if mortality is falling. There are quantitative differences from the constant-fertility case, however. If TFR rises to replacement level gradually rather than instantaneously, the time that the population takes to "recover" from negative momentum is almost twice as long.

3.4.4 Short-Run Effects on Age Structure

Age structure is probably more important than population size in the political arena, because many social welfare programs – old-age pensions in particular – only function well with a sufficiently favorable distribution of population by age. In this section we analyze the age structure changes that would occur if a country followed the stationary fertility paths in a

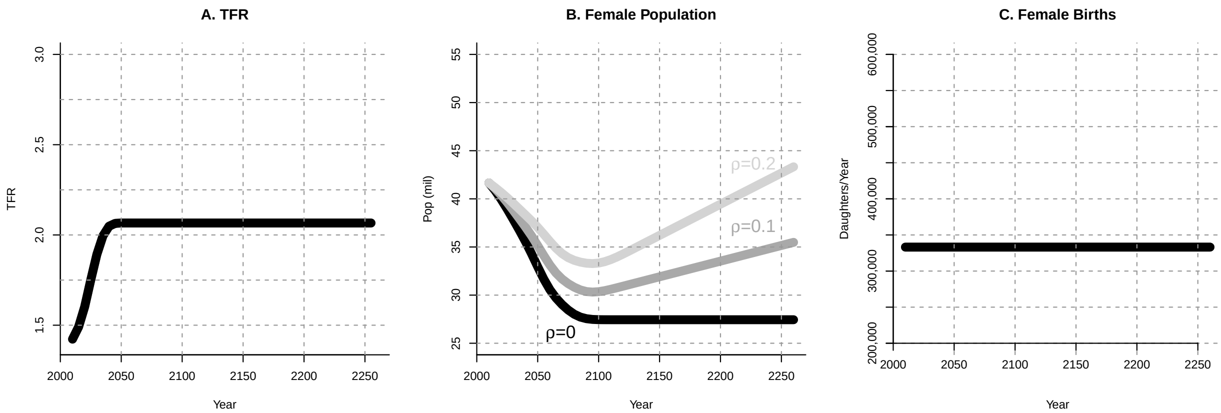

Figure 8: Constant-Birth Trajectory, with alternative rates of delayed senescence and decreasing adult mortality. See Figure 3 caption for more details. Nearly universal maternal survival means that the three separate lines in Panel A overlap almost completely.

world of continually decreasing mortality.

We use a conventional index of age structure, the support ratio $SR = \frac{N_{15-64}}{N_{65+}}$, which approximates the number of workers per retiree. This measure, also used by the UN (2001) in its analysis of replacement migration, is the reciprocal of the widely-used old-age dependency ratio. Higher SR values correspond to younger (adult) age structures.

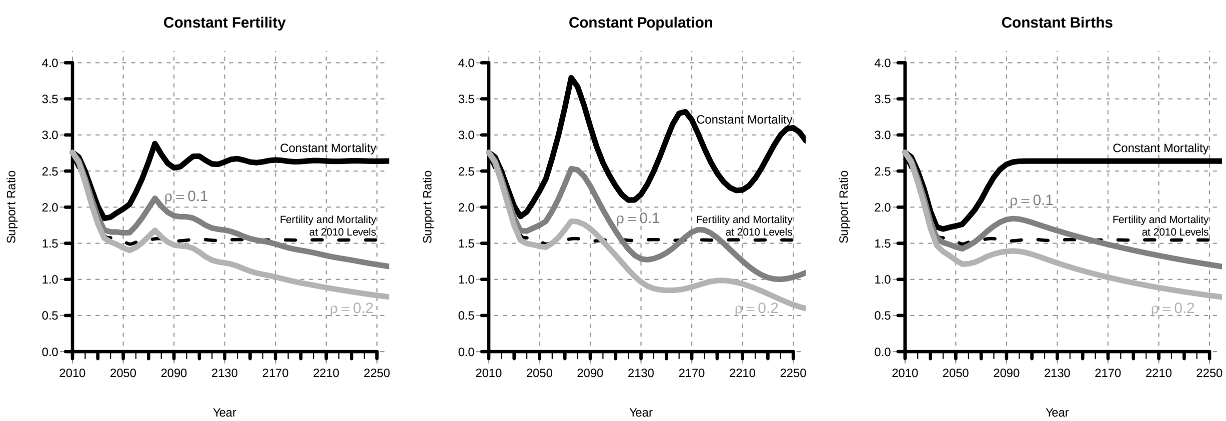

Figure 9: Evolution of the support ratio (workers per retiree) under alternative fertility paths and mortality changes. Each panel represents an alternative fertility policy. Solid lines represent different rates of senescence delay; dashed line represents the time path of the support ratio in the shrinking population implied by constant vital rates. As shown in Table 2, if $\rho > 0$ (grey curves) the support ratio has a limiting value of 0.01 as $t \to \infty$.

As shown in Figure 9, Germany had a support ratio of 2.76 in 2010, which would eventually decline to 1.55 if 2010 vital rates continued. Consider first the cases in which *future mortality remains constant*, represented by dark solid lines in each panel. Without mortality change, any of the three stationary fertility policies produces SR=2.59 in the long run. Convergence to SR=2.59 would occur within about one lifespan under constant-fertility or constant-birth policies (first and third panels), but would be extremely slow under a constant-population policy. Furthermore, a constant-population policy (middle panel) would generate large fluctuations in the support ratio for many hundreds of years.

Now consider the cases in which *future mortality falls due to delayed senescence*, represented by grey lines in each panel. Darker and lighter grey lines correspond to ρ values of 0.1 and 0.2, respectively. From theoretical results in Table 2 we know that the long-run limiting values of SR are extraordinarily low (0.01, or 1 worker per 100 retirees) under delayed senescence. In contrast, the grey lines in Figure 9 illustrate the shorter-term consequences of decreasing mortality for stationary fertility policies. Comparing the black and grey lines shows that, in all cases, 40–50 years of delayed senescence and increasing life expectancy would affect the support ratio significantly. In other words, mortality improvements at older ages could quickly cancel the rejuvenating effects of stationary fertility policies. From comparison of the dashed and grey lines, it is also clear that a century or more of mortality decline, even with stationary fertility policies, could lead to support ratios below the unfavorable stable values implied by current vital rates.

For policy makers, the most interesting (and probably frustrating) pattern in Figure 9 is the almost complete overlap of all SR trajectories over the first 20 years of projections. In a closed population, the near-term future of the support ratio is predetermined – irrespective of any developments in fertility and mortality. Fertility change cannot logically alter the number of workers until at least 15 years later, and plausible patterns of mortality change would have only very small effects on the support ratio in the first several decades. Thus, for Germany, any changes in vital rates would not have any practical relevance for the support ratio until the mid-2030s.

4 Conclusion

In this paper we analyze one familiar and two lesser-known fertility policies that lead to zero population growth if age-specific mortality is constant. We also investigate the long-run implications of the same fertility trajectories when the mortality schedule is *not* constant.

Surprisingly, we find that with one kind of never-ending mortality change – continually-delayed senescence – classic stationary policies *also* lead to zero population growth in the long run, although with a very old and unfamiliar asymptotic age structure. This is a new kind of stationary population, in which fertility rates must converge to constant levels, but the mortality schedule never stops changing.

In order to understand how, when, and by how much improved old-age survival might offset the expected stabilizing and rejuvenating effects of classic stationary policies, we also simulated short- and medium-run trajectories. Here the news is not so good. Negative population momentum is a powerful force. It can only be eliminated through unrealistically high variability in future fertility rates and age structures. On more realistic demographic paths, built-in population decline would be smaller if old-age survival increased, but declines would nonetheless continue for at least a century.

Populations pursuing stationary fertility policies would age significantly if survival steadily improved. In any scenario, fertility change cannot logically affect the inevitably declining worker/retiree balance for several decades. Futhermore, in all simulations (Figure 9) the aging effects of delayed senescence quickly lead to much lower support ratios in the decades that follow that initial dip. Thus, even in the short to medium run, mortality change is likely to cancel out the rejuvenating effects of stationary fertility.

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