PAA 2015 draft paper: The sensitivity analysis of population projections: models structured by age and sex*

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September 26, 2014

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1 Introduction

Fifty years ago, in the first issue of the first volume of the then-new journal *Demography*, Nathan Keyfitz (1964) described the "population projection as a matrix operator." He showed that population projections using the cohort component method could be written as matrix population models, and emphasized the value in doing so to focus attention on the mathematical structure of the projection, inviting deeper analyses of its properties with more powerful mathematical tools. Today, official projections are often implemented as computer algorithms, the details of which are obscure but which permit almost endless fine-tuning of relationships. But the advantages of considering projections as matrix operators are no less real. In this paper, we carry on in this spirit, using matrix calculus methods to develop a complete perturbation analysis of population projections.

As is customary in demography, we use the term *projection* to describe a conditional prediction of population size and structure, over a specified time horizon, such as are regularly developed by national governments, international consortia (e.g., Eurostat), and non-governmental organizations (U.N.). All projections are conditional in the sense that they are based on one or more hypothetical scenarios defining future rates of mortality, fertility, and migration (collectively, the "vital rates"), and also conditional on an initial population.

The vital rate scenarios are defined in terms of a set of parameters; the nature of those parameters will depend on the details of the scenarios. Sensitivity analysis (also called perturbation analysis) asks how the results of the projection would change in response to changes in the parameters. Sensitivity analysis is useful because:

1. It can project the consequences of changes in the vital rates. Such changes could result from human actions, either intentional (e.g., policies to encourage reproduction, public health interventions, or conservation strategies applied to endangered species) or unintentional (e.g., consequences of pollution or environmental degredation), or natural changes.

2. It can be used to compare potential policy interventions and identify interventions that would have particularly large effects. If an outcome is particularly sensitive to a particular parameter, that parameter may be an attractive target for intervention.

3. It can be used retrospectively to decompose observed changes in some outcome into contributions from changes in each of the parameters (Caswell 2000, 2001).

4. It can be used to identify parameters the estimation of which deserves extra attention, because they have large effects on the results.

5. It can quantify uncertainty of projection results: given the uncertainty in some parameter θ, and the sensitivity of an outcome of interest to changes in θ, it is possible to approximate the resulting uncertainty in the outcome. Demogaphers have become increasingly concerned with estimating the uncertainty of projection results (Booth 2006, Ahlburg and Lutz 1998).

1.1 Sensitivity and elasticity

Our approach is to calculate the derivatives of the projection results to the parameters and initial conditions. This gives the effects of small changes, gives approximate results for quite large changes, and identifies parameters with particularly large or small impacts on the results. As we will show, the parameters may include aspects of mortality, fertility, or immigration. The projection results may include a variety of different functions of the population, including measures of size, structure, and growth.

We will present results for both sensitivity and elasticity. If y is a function of x, we define the sensitivity of y to changes in x as

$$\text{sensitivity} = \frac{dy}{dx}. \tag{1}$$

The elasticity of y is the proportional sensitivity, which is

$$\begin{aligned} \text{elasticity} &= \frac{x}{y}\frac{dy}{dx} && (2) \\ &= \frac{\epsilon y}{\epsilon x} && (3) \end{aligned}$$

This gives the proportional change in y resulting from a proportional change in x. There is no standard notation for elasticities, despite their widespread use in economics and population biology. The notation used here, $\epsilon y / \epsilon x$, which parallels the notation for derivatives, is adapted from a notation used by Samuelson (1947). Elasticities are only defined when $y > 0$ and $x \geq 0$.

In Section 2 we will write both one-sex and two-sex projections as matrix operators, and discuss the scenarios that might be involved in such projections and the parameters that might determine those scenarios. Then, in Section 3 we will give the expressions for the sensitivities and elasticities of the population vector (abundance by age class of males, or females, or both combined) to changes in mortality, fertility, and immigration. A particularly important part of our results, in Section 3.5, is to show how the sensitivity results for the population vector can be translated directly into other dependent variables, such as weighted population size, ratios, and growth rates.

Our approach here is to write the projection as a matrix operator, and then to use matrix calculus (e.g., Caswell 2007, 2008, 2012) to derive the needed derivatives of the results to underlying parameters. These methods are easily implemented in any matrix-oriented computer language, especially Matlab, but also R.

After presenting the theory, in Section 4 we will apply the calculations to a projection of the population of Spain, using information from the Instituto Nacional de Estadistica (INE). We conclude with a discussion of how these results apply to evaluating the uncertainty of projections and future developments.

Notation. Matrices are denoted by upper case bold symbols (e.g., $\mathbf{A}$) and vectors by lower case bold symbols (e.g., $\mathbf{n}$). All vectors are column vectors by default. The vector $\mathbf{x}^{\mathsf{T}}$ is the transpose of the vector $\mathbf{x}$. The Hadamard, or element-by-element, product of $\mathbf{A}$ and $\mathbf{B}$ is $\mathbf{A} \circ \mathbf{B}$. The Kronecker product is $\mathbf{A} \otimes \mathbf{B}$. The diagonalization operator $\mathcal{D}(\mathbf{x})$ creates a matrix with $\mathbf{x}$ on the diagonal and zeros elsewhere. The vec operator, when applied to a $m \times n$ matrix $\mathbf{X}$ creates a $mn \times 1$ vector $\operatorname{vec} \mathbf{X}$ by stacking each column of $\mathbf{X}$ on top of the next. When necessary, subscripts are attached to indicate the size of matrices or vectors; e.g., $\mathbf{I}_s$ is the $s \times s$ identity matrix. The vector $\mathbf{1}$ is a vector of ones, and the vector $\mathbf{e}_i$ is the ith unit vector, with a 1 in the ith location and zeros elsewhere.

2 Projection as a matrix operation

2.1 Dynamics

Any cohort-component population projection can be written as a matrix operator. As a simple example, we present a one-sex model, but we focus most of our attention on a two-sex model that includes separate rates for males and females. Multistate projections will be considered in a subsequent paper.

A single-sex projection can be written as

$$\mathbf{n}(t+1) = \mathbf{A}(t)\mathbf{n}(t) + \mathbf{b}(t) \qquad \mathbf{n}(0) = \mathbf{n}_0 \tag{4}$$

where $\mathbf{n}(t)$ is a vector whose entries are the numbers of individuals in each age class or stage at time t, $\mathbf{A}(t)$ is a projection matrix incorporating the vital rates at time t, and $\mathbf{b}(t)$ is a vector giving the number of immigrants in each age class or stage at time t. The projection begins with a specified initial condition, denoted $\mathbf{n}_0$, and is carried out until some target time T.

Two-sex projections are generalizations of (4). We define population vectors $\mathbf{n}_f$ and $\mathbf{n}_m$, and projection matrices $\mathbf{A}_f$ and $\mathbf{A}_m$, for females and males, respectively. We assume that reproduction is female dominant[1], so all fertility is attributed to females. We decompose the projection matrices for females and males into

$$\mathbf{A}_f(t) = \mathbf{U}_f(t) + r\mathbf{F}(t) \tag{5}$$

$$\mathbf{A}_m(t) = \mathbf{U}_m(t) \tag{6}$$

[1]Two-sex models that do not assume dominance by one sex have been used to project animal populations, but not, as far as we know, human populations (Jenouvrier et al. 2010, 2012, 2014).

where $\mathbf{U}$ describes transitions and survival of extant individuals and $\mathbf{F}$ describes the production of new individuals by reproduction.

In an age-classified model, $\mathbf{F}$ will have fertilities on the first row and zeros elsewhere. A proportion r of the offspring are **female**. This model attributes reproduction to females; hence there is no need to create separate fertility matrices for reproduction by males and females.

The male component of the population is projected by the survival matrix $\mathbf{U}_m$; the input of new individuals comes from the female population. The projection model becomes

$$\mathbf{n}_f(t+1) = \left[\mathbf{U}_f(t) + r\mathbf{F}(t)\right]\mathbf{n}_f(t) + \mathbf{b}_f(t) \tag{7}$$

$$\mathbf{n}_m(t+1) = \mathbf{U}_m(t)\mathbf{n}_m(t) + (1-r)\mathbf{F}(t)\mathbf{n}_f(t) + \mathbf{b}_m(t) \tag{8}$$

The formulations (4) and (7)–(8) are general enough to encompass all the projections typically used. The vector $\mathbf{n}$ can incorporate any type of population structure considered relevant. If individuals are grouped into age classes, then $\mathbf{A}$ is the familiar Leslie matrix, with survival probabilities on the subdiagonal, fertilities in the first row, and zeros elsewhere. If individuals are classified by other criteria ("stages" in common usage), $\mathbf{A}$ will have the structure needed to capture transitions among stages based on physiological condition, developmental stage, socio-economic grouping, marital status, parity status, etc.

Immigration, denoted here by $\mathbf{b}(t)$, is a particularly challenging part of population projection. We explore the reasons for this, and some of the ways in which migration is handled, in Section 6.3. Some implementations of migration require minor modifications of equations (4)–(8), but the sensitivities are derived in the same way as what we are about to show.

2.2 Scenarios and parameters

A projection is based on a scenario of how the future might unfold. The matrices $\mathbf{U}(t)$ and $\mathbf{F}(t)$, and the vector $\mathbf{b}(t)$, describe the future dynamics of the mortality, fertility, and immigration. The future being unknown, considerable ingenuity is required to construct these functions. Three major approaches seem to be used, singly or in combination.

1. Extrapolation of trends. This approach starts from the observation that some vital rates (particularly mortality and fertility rates) develop gradually over time, and extrapolates

those patterns into the future. The best-known of these is perhaps the Lee-Carter model for mortality, which projects mortality with a time-series model applied to a singular value decomposition of a past record of age- and time-specific mortality rates. Recent developments include sophisticated Bayesian methods that also produce statistically rigorous uncertainty bounds (e.g., Gerland et al. 2014).

2. Assumptions and expert opinion. Future trends in vital rates are sometimes simply assumed, based on unspecified conceptual models. The projections of Eurozone countries by Eurostat, for example, are based on the assumption that the mortality and fertility of all European countries will converge to a common value by the year 2150 (Lanzieri 2009). The rates for a given country in each year are determined by interpolating between the rates at the start of the projection and the final target rates. Other studies have been based on the opinion of experts who are not directly involved in the projection process. Lutz and colleagues, for instance, have used a Delphi-method based approach to collect and aggregate external expert opinions on demographic trends in a systematic manner (Ahlburg and Lutz 1998). Expectations of population members about their own lives (e.g. survey data on the expected number of children or expected remaining life expectancy) have also been used to define scenarios.

3. Dependence on external factors, which can themselves be projected. If the vital rates depend on some factor, and the dynamics of that factor can be predicted, this provides the basis for a projection of the vital rates. The appoach has been used for animal populations. For example, projections of populations of polar bears and emperor penguins under the impact of climate change have been based on projections of sea ice conditions (a critical environmental variable for these species) generated by models of global climate conditions produced by the IPCC (Hunter et al. 2010, Jenouvrier et al. 2009, 2012, 2014). Similarly, projections of human populations have been based on expectations about future economic, social or environmental developments (Booth 2006).

Regardless of how the scenario of future conditions is obtained, the resulting projection depends on a set of *parameters* which jointly determine the projection matrices and the immigration vectors. We will write this set of parameters as a vector $\boldsymbol{\theta}$, of dimension p. In this paper, we focus on the commonly encountered case in which the parameters are the age- and time-specific rates of mortality, fertility, and immigration:

$$\boldsymbol{\theta}(t) = \begin{cases} \boldsymbol{\mu}(t) & \text{vector of mortality rates} \\ \mathbf{f}(t) & \text{vector of age-specific fertility} \\ \mathbf{b}(t) & \text{immigration vector} \end{cases} \tag{9}$$

These vectors might, in turn, be expressed as functions of a scalar quantity such as life expectancy, or a parametric model such as the Gompertz, gamma-Gompertz, or Siler models for mortality, or the Coale-Trussel function for fertility. In that case, the vector $\boldsymbol{\theta}$ would include the parameters that define those functions.

3 Perturbation analysis of projections

Our goal is to quantify the sensitivity and elasticity of projection results to the parameters in $\boldsymbol{\theta}$. To do that, we need to introduce the matrix calculus framework for derivatives of vectors (the projection output) with respect to other vectors (the parameter vector).

3.1 Matrix calculus notation

Matrix calculus permits the differentiation of scalar-, vector-, or matrix-valued functions of scalar-, vector-, or matrix-valued arguments.

The underlying theory is developed in detail by Magnus and Neudecker (1987); for an introductory account see Abadir and Magnus (2005). The methods have been applied to demography in a series of papers (Caswell 2006, 2007, 2008, 2010, 2011, 2012, Caswell and Shyu 2012, van Raalte and Caswell 2013, Engelman et al. 2014).

If $\mathbf{y}$ is a $n \times 1$ vector function of the $m \times 1$ vector $\mathbf{x}$, then the sensitivity of $\mathbf{y}$ to $\mathbf{x}$ is the $n \times m$ Jacobian matrix written as

$$\frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}} = \left(\frac{dy_i}{dx_j} \right). \tag{10}$$

We will use the fact that this calculus satisfies the chain rule, so that if $\mathbf{z}$ is a function of $\mathbf{y}$, then

$$\frac{d\mathbf{z}}{d\mathbf{x}^{\mathsf{T}}} = \frac{d\mathbf{z}}{d\mathbf{y}^{\mathsf{T}}} \frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}}. \tag{11}$$

The elasticity of $\mathbf{y}$ is the $n \times m$ matrix given by

$$\frac{\epsilon\mathbf{y}}{\epsilon\mathbf{x}^{\mathsf{T}}} = \mathcal{D}(\mathbf{y})^{-1} \left(\frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}} \right) \mathcal{D}(\mathbf{x}) \tag{12}$$

Our goal is to obtain a set of sensitivity and elasticity relationships of the form

$$\frac{d\boldsymbol{\xi}}{d\boldsymbol{\theta}^{\mathsf{T}}} \quad \text{and} \quad \frac{\epsilon\boldsymbol{\xi}}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}}$$

where $\boldsymbol{\xi}$ is a projection output. This output might be $\mathbf{n}(t)$, the population vector, or it might be some scalar function of $\mathbf{n}$ (e.g., a dependency ratio).

In each case the sensitivity is obtained from a dynamic model for the derivative

$$\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}(x)}$$

If there are ω age classes and p parameters, then this derivative is a $\omega \times p$ matrix whose (i, j) entry is the derivative of $n_i(t)$ with respect to the parameter θ_i.

3.2 One-sex projections

For simplicity, we begin with the one-sex projection (4). We consider the effects of changes in the parameters at time x on the projected population at time t, for $x = 0, \ldots, T$ and $t = 0, \ldots, T$. Changes in $\boldsymbol{\theta}(x)$ obviously have no effect on $\mathbf{n}(t)$ for $t < x$ (we ignore the complications of time travel). However, a perturbation at time x will ripple through $\mathbf{n}(t)$ for all $t > x$, and our goal is to find out how.

The dynamics of the population vector $\mathbf{n}(t)$ are obtained by iterating equation (4). The sensitivity of $\mathbf{n}(t)$ to a change in $\boldsymbol{\theta}(x)$ is obtained by iterating the dynamic equation

$$\frac{d\mathbf{n}(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{A}(t) \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \left(\mathbf{n}^{\mathsf{T}}(t) \otimes \mathbf{I} \right) \frac{d\text{vec}\,\mathbf{A}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{13}$$

starting from the initial condition

$$\frac{d\mathbf{n}(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{0}_{\omega\times p} \tag{14}$$

The elasticity of $\mathbf{n}(t)$ to $\boldsymbol{\theta}(x)$ is, from (12),

$$\frac{\epsilon\mathbf{n}(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathcal{D}\Big(\mathbf{n}(t)\Big)^{-1}\;\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}\;\mathcal{D}\left[\boldsymbol{\theta}(x)\right] \tag{15}$$

The structure of (13) is common to all the sensitivity results:

$$\underbrace{\frac{d\mathbf{n}(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{sensitivity at } t+1} = \underbrace{\mathbf{A}(t)\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{sensitivity at } t} + \underbrace{\left(\mathbf{n}^{\mathsf{T}}(t)\otimes\mathbf{I}\right)\frac{d\mathrm{vec}\,\mathbf{A}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{effects via } \mathbf{A}} + \underbrace{\frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{effects via } \mathbf{b}} \tag{16}$$

The sensitivity at $t+1$ is projected from the sensitivity at t, the effects of parameters on the projection matrix, and the effects of parameters on the immigration vector.

3.3 Two-sex projections

The sensitivity of the two-sex projection is given by the two derivatives,

$$\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \quad \text{and} \quad \frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}$$

These derivatives are obtained from dynamic expressions, for the female population

$$\begin{aligned}\frac{d\mathbf{n}_f(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} &= \Big(\mathbf{U}_F(t)+r\mathbf{F}(t)\Big)\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}\right)\left(\frac{d\mathrm{vec}\,\mathbf{U}_F(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + r\frac{d\mathrm{vec}\,\mathbf{F}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}\right) \\ &\quad + \frac{d\mathbf{b}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}\end{aligned} \tag{17}$$

and the male population

$$\underbrace{\frac{d\mathbf{n}_m(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{sensitivity at } t+1} = \underbrace{\mathbf{U}_m(t)\frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + (1-r)\mathbf{F}(t)\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{sensitivities at } t} + \underbrace{\left(\mathbf{n}_m^{\mathsf{T}}(t)\otimes\mathbf{I}\right)\frac{d\mathrm{vec}\,\mathbf{U}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{effects via male transitions}}$$

$$+\underbrace{(1-r)\left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}\right)\frac{d\mathrm{vec}\,\mathbf{F}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{effects via female fertility}} + \underbrace{\frac{d\mathbf{b}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}}_{\text{effects via immigration}} \tag{18}$$

Equations (17) and (18) are iterated from initial conditions

$$\frac{d\mathbf{n}_f(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{d\mathbf{n}_m(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{0}_{\omega\times p} \tag{19}$$

along with the iteration of equations (7) and (8) for the population vectors $\mathbf{n}_f(t)$ and $\mathbf{n}_m(t)$.

We have labelled the terms in (18) to show the parallels with (16). In both cases, the sensitivity at time $t+1$ depends on the sensitivity at time t and on the effects of the parameter vector on the transition and fertility matrices and on the immigration vector. In the next section we turn to the calculation of these derivatives.

The elasticities of $\mathbf{n}_f(t)$ and $\mathbf{n}_m(t)$ are given by applying (15) to the corresponding derivatives for female and male population:

$$\frac{\epsilon\mathbf{n}_f(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathcal{D}\left[\mathbf{n}_f(t)\right]^{-1}\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}\,\mathcal{D}[\boldsymbol{\theta}(x)] \tag{20}$$

and similarly for $\mathbf{n}_m$.

The combined population of both males and females is $\mathbf{n}_c = \mathbf{n}_f + \mathbf{n}_m$. The sensitivity and elasticity of $\mathbf{n}_c$ are

$$\frac{d\mathbf{n}_c(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{21}$$

$$\frac{\epsilon\mathbf{n}_c(t)}{\epsilon\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathcal{D}\left[\mathbf{n}_c(t)\right]^{-1}\left[\frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}\right]\mathcal{D}[\boldsymbol{\theta}(x)] \tag{22}$$

The entire system of sensitivity and elasticity relationships is obtained by simultaneously iterating equations (7) and (8) to project the populations of females and males, and the equations (17) and (18) to project the sensitivity of the female and male populations.

3.4 Parameters and the derivatives of matrices

So far we have left the parameter vector $\boldsymbol{\theta}$ undefined, because the results apply to any choice of parameter. Now we become more specific by focusing on the cases where $\boldsymbol{\theta}$ is a vector of mortality rates, or of fertilities, or of immigration rates. We consider each of these important cases and present the derivatives of the matrices $\mathbf{U}$ and $\mathbf{F}$, and the vector $\mathbf{b}$, to those parameters. These derivatives appear in the expressions (17), (18), and (21) and the corresponding elasticity equations.

A change in the parameter vector $\boldsymbol{\theta}$ at time x can affect the projection matrices only when $t = x$; to indicate this, we will use the Kronecker delta function

$$\delta(x,t) = \begin{cases} 1 & \text{if } x = t \\ 0 & \text{if } x \neq t \end{cases} \tag{23}$$

Because sex-specific mortality only affects the matrices for that sex, the following results apply to either male or female rates, so we do not include the subscript to define the sex of the subpopulation.

- **Mortality:** $\boldsymbol{\theta} = \boldsymbol{\mu}$. Mortality rates affect the transition matrix $\mathbf{U}$ (or the projection matrix $\mathbf{A}$ if transitions and fertility are not separated). Define the survival vector $\mathbf{p} = \exp(-\boldsymbol{\mu})$, which appears on the subdiagonal of $\mathbf{U}$, and an indicator matrix $\mathbf{Z}$ with ones on the subdiagonal and zeros elsewhere. Then

$$\frac{d\text{vec}\,\mathbf{A}(t)}{d\boldsymbol{\mu}^{\mathsf{T}}(x)} = \frac{d\text{vec}\,\mathbf{U}(t)}{d\boldsymbol{\mu}^{\mathsf{T}}(x)} = -\delta(x,t)\mathcal{D}(\text{vec}\,\mathbf{Z})\left(\mathbf{1}\otimes\mathbf{I}\right)\mathcal{D}\left(\mathbf{p}(t)\right) \tag{24}$$

  where $\mathbf{1}$ is a vector of ones. The derivatives of $\mathbf{F}$ and $\mathbf{b}$ with respect to $\boldsymbol{\mu}$ are zero.

- **Fertility:** $\boldsymbol{\theta} = \mathbf{f}$. The fertility vector appears on the first row of the matrix $\mathbf{F}$. The derivative of $\mathbf{F}$ is

$$\frac{d\text{vec}\,\mathbf{F}(t)}{d\mathbf{f}^{\mathsf{T}}} = \delta(x,t)\left(\mathbf{I}\otimes\mathbf{e}_1\right) \tag{25}$$

  where $\mathbf{e}_1$ is the first unit vector. The derivatives of $\mathbf{U}$ and $\mathbf{b}$ with respect to $\mathbf{f}$ are zero.

- **Immigration:** $\boldsymbol{\theta} = \mathbf{b}$. When the parameter vector is the immigration vector, then

$$\frac{d\mathbf{b}(t)}{d\mathbf{b}^{\mathsf{T}}(x)} = \delta(x,t)\mathbf{I} \tag{26}$$

and the derivatives of $\mathbf{U}$, $\mathbf{F}$, and $\mathbf{A}$ with respect to $\mathbf{b}$ are all zero.

3.5 Choosing a dependent variable

These results presented so far provide the sensitivity of *every* age class, at *every* time from 0 to T, with respect to changes in mortality, fertility, and immigration of *every* age class, at *every* time from 0 to T. This high-dimensional structure is more information than anyone wants, but it can be condensed to provide information on the sensitivity of any projection outcome that is of interest. An informal survey of Statistical Offices[2] finds that they typically present projections of the total population size, the proportional representation of specific age groups (e.g., working age adults, school-age children, people of retirement age, women of childbearing age), ratios such as the old-age, young-age, and total dependency ratios, and descriptors of the age distribution such as the median age in the population.

In this section, we show how to calculate the sensitivity and elasticity of such dependent variables from the derivatives of $\mathbf{n}(t)$ given in (17), (18), and (21). In the following, sensitivities can be applied to the female population, the male population, or the combined population.

1. Total population size $N(t)$. The total population size is $N(t) = \mathbf{1}^\mathsf{T}\mathbf{n}(t)$; its sensitivity to parameter changes at time x is
$$\frac{dN(t)}{d\boldsymbol{\theta}^\mathsf{T}(x)} = \mathbf{1}^\mathsf{T}\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^\mathsf{T}(x)} \tag{27}$$
The elasticity of $N(t)$ is
$$\frac{\epsilon N(t)}{\epsilon\boldsymbol{\theta}^\mathsf{T}(x)} = \frac{1}{N(t)}\frac{dN(t)}{d\boldsymbol{\theta}^\mathsf{T}(x)}\,\mathcal{D}(\boldsymbol{\theta}) \tag{28}$$

2. *Weighted* total population size. Suppose that $N(t) = \mathbf{c}^\mathsf{T}\mathbf{n}(t)$, where $\mathbf{c}$ is a vector that applies different weights to each age class. For example, $\mathbf{c}$ might contain the labor income of each age class, or the prevalence in each age class of some health condition. $N(t)$ is now a weighted population size; the sensitivity of $N(t)$ to a change in parameters at time x is
$$\frac{dN(t)}{d\boldsymbol{\theta}^\mathsf{T}(x)} = \mathbf{c}^\mathsf{T}\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^\mathsf{T}(x)}. \tag{29}$$

[2] European Union, Germany, France, Belgium, Ireland, Estonia, Spain, Austria, Finland, Sweden, United Kingdom, Iceland, and Switzerland

The elasticity is again given by (28).

The weight vector $\mathbf{c}$ might also subject to perturbations (e.g., if the prevalence of a health condition was to change by screening or treatment). The sensitivity of $N(t)$ to changes in $\mathbf{c}$ is

$$\frac{dN(t)}{d\mathbf{c}^{\mathsf{T}}} = \mathbf{n}^{\mathsf{T}}(t) \tag{30}$$

The corresponding elasticities of $N(t)$ to $\boldsymbol{\theta}$ and $\mathbf{c}$ are

$$\frac{\epsilon N(t)}{\epsilon \mathbf{c}^{\mathsf{T}}} = \frac{1}{N(t)} \mathbf{n}^{\mathsf{T}}(t) \mathcal{D}(\mathbf{c}) \tag{31}$$

The elasticities of $N(t)$ to $\mathbf{c}$ in (31) always sum to 1.

3. Ratios of weighted population sizes. Let

$$R(t) = \frac{\mathbf{a}^{\mathsf{T}}\mathbf{n}(t)}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)}, \tag{32}$$

where $\mathbf{a}$ and $\mathbf{c}$ are vectors of weights. Such ratios appear frequently as dependent variables in population projections. Examples of include:

   (a) The proportional representation of an age group (e.g., the proportion over 65 years of age). In this case, $\mathbf{a}$ is an indicator vector, containing ones corresponding to the ages in the age group, and zeros elsehwere. The vector $\mathbf{c} = \mathbf{1}$, so that $\mathbf{c}^{\mathsf{T}}\mathbf{N}$ is the total population size.

   (b) Dependency ratios. In this case, $\mathbf{a}$ and $\mathbf{c}$ are both indicator vectors for the relevant age groups. The old-age dependency ratio, for example, is obtained by letting $\mathbf{a}$ indicate ages beyond retirement age and $\mathbf{c}$ indicate working ages.

   (c) Weighted dependency ratios. Instead of considering all individuals of retirement age, or working age, to be equal, $\mathbf{a}$ and $\mathbf{c}$ can be vectors of weights. For example, the economic support ratio (Prskawetz and Sambt 2014) is computed by letting $\mathbf{a}$ be a vector giving age-specific labor income, and $\mathbf{c}$ a vector giving age-specific consumption.

   (d) Moments of the age distribution. The mean of the age distribution is obtained by setting the vector **a** to the midpoints of the age intervals; e.g., for one year age classes,

$$\mathbf{a} = \begin{pmatrix} 0.5 & 1.5 & 2.5 & \cdots \end{pmatrix}^{\mathsf{T}} \tag{33}$$

and setting $\mathbf{c} = \mathbf{1}$. The second moment of the age distribution is obtained by setting

$$\mathbf{a} = \begin{pmatrix} 0.5^2 & 1.5^2 & 2.5^2 & \cdots \end{pmatrix}^{\mathsf{T}} \tag{34}$$

and $\mathbf{c} = 1$. The variance in age is obtained from the first and second moments in the usual way.

(e) Moments of age-specific properties. Suppose that $B(x)$ is some measurement on age class x (e.g., the mean body mass index (BMI) of age class x). Then the mean BMI in the population would be obtained by setting $\mathbf{c} = \mathbf{1}$ and

$$\mathbf{a} = \begin{pmatrix} B(1) & B(2) & B(3) & \cdots \end{pmatrix}^{\mathsf{T}}. \tag{35}$$

The sensitivity of a ratio (Caswell 2007) is

$$\frac{dR(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{dR(t)}{d\mathbf{n}^{\mathsf{T}}(t)} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{36}$$

$$= \left(\frac{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\mathbf{a}^{\mathsf{T}} - \mathbf{a}^{\mathsf{T}}\mathbf{n}(t)\mathbf{c}^{\mathsf{T}}}{\left(\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\right)^2} \right) \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}. \tag{37}$$

The elasticity of the ratio is

$$\frac{\epsilon R(t)}{\epsilon \boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{1}{R(t)} \frac{dR(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \mathcal{D}\left[\boldsymbol{\theta}(x)\right] \tag{38}$$

4. Short-term growth rates. Define the k-step growth rate of the weighted population size $\mathbf{c}^{\mathsf{T}}\mathbf{n}$, at time t as

$$\lambda(t) = \frac{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)}. \tag{39}$$

This gives the average growth rate of the population over the next k years, starting from year

t. To obtain the sensitivity of $\lambda(t)$, note that

$$\frac{d\lambda(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{\partial\lambda(t)}{\partial\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)}\frac{d\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{\partial\lambda(t)}{\partial\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}\frac{d\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{40}$$

From (39), we have

$$\frac{\partial\lambda(t)}{\partial\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} = \frac{-\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}{\left[\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\right]^2} \tag{41}$$

$$\frac{\partial\lambda(t)}{\partial\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)} = \frac{1}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)} \tag{42}$$

Assembling all the pieces gives the sensitivity of the short-term k-step growth rate,

$$\frac{d\lambda(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{-\mathbf{c}^{\mathsf{T}}\mathbf{n}(t+k)}{\left[\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)\right]^2}\mathbf{c}^{\mathsf{T}}\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{1}{\mathbf{c}^{\mathsf{T}}\mathbf{n}(t)}\mathbf{c}^{\mathsf{T}}\frac{d\mathbf{n}(t+k)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{43}$$

In the special case where interest focuses on total population size, one simply sets $\mathbf{c} = \mathbf{1}$.

The quantity λ is a discrete time growth rate; the corresponding continuous growth rate over the interval is given by $r(t) = \log(\lambda(t))/k$, and

$$\frac{dr(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \frac{1}{k\lambda(t)}\frac{d\lambda(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{44}$$

3.6 Aggregating perturbations over age and time

The expressions presented so far give the response of every age class in the population $\mathbf{n}$, at any time t, to a perturbation of any of the parameters in $\boldsymbol{\theta}$, at any other time x. This is a 4-dimensional information structure, and it will often be appropriate to simplify the structure by aggregating sensitivity over age, or time, or parameters, or all of these. Some examples are:

1. The sensitivity of $\mathbf{n}$ at time t to a perturbation, at time x, that affects all age classes by the same amount (e.g., an additive or a proportional hazard imposed on the mortality schedule). The sensitivity and elasticity are given by

$$\text{sensitivity:} \qquad \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}\mathbf{1} \tag{45}$$

$$\text{elasticity:} \qquad \frac{\epsilon \mathbf{n}(t)}{\epsilon \boldsymbol{\theta}^{\mathsf{T}}(x)} \mathbf{1} \tag{46}$$

2. The sensitivity of the population vector at time t to a change in $\boldsymbol{\theta}(x)$ that is applied equally at every time from $x = 0$ to $x = T$. In a slight abuse of notation, let us denote the sensitivity of $\mathbf{n}(t)$ to this perturbation as

$$\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(0,T)} = \sum_{x=0}^{T} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{47}$$

The corresponding elasticity is

$$\frac{\epsilon \mathbf{n}(t)}{\epsilon \boldsymbol{\theta}^{\mathsf{T}}(0,T)} = \mathcal{D}[\mathbf{n}(t)]^{-1} \sum_{x=0}^{T} \left(\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \mathcal{D}[\boldsymbol{\theta}(x)] \right) \tag{48}$$

$$= \sum_{x=0}^{T} \frac{\epsilon \mathbf{n}(t)}{\epsilon \boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{49}$$

3. The response of a summation of population properties over time. For example, consider the the population vector summed from time $t = 0$ to $t = T$. The sensitivity and elasticity of this sum are

$$\frac{d}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \sum_{t=0}^{T} \mathbf{n}(t) = \sum_{t=0}^{T} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{50}$$

$$\frac{\epsilon}{\epsilon \boldsymbol{\theta}^{\mathsf{T}}(x)} \sum_{i=0}^{T} \mathbf{n}(t) = \mathcal{D}\left[\sum_{t} \mathbf{n}(t) \right]^{-1} \sum_{t=0}^{T} \frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \mathcal{D}\left[\boldsymbol{\theta}(x)\right] \tag{51}$$

4 Projection of the population of Spain

To illustrate the use of matrix calculus techniques for sensitivity and elasticity calculations, we use a projection of the population of Spain, published by the Spanish Instituto Nacional de Estadistica (INE). The projection uses the cohort component method and distinguishes single-year age groups (ages 0 to 100+ years) and sex of population members. It covers the years 2012 to 2052. Projection intervals have the length of one year (INE 2012a). The projection is based on the following assumptions:

- The fertility scenario is presented in the form of age-specific fertility rates. INE assumes that the total fertility rate will increase from 1.36 children per women in 2011 to 1.56 in 2051, and that the mean age at childbearing will rise from 31 to 32 years within the same period. On their internet webpages, INE has published fertility vectors for $\mathbf{f}(t)$ for $t = 1, \ldots, 40$ which reflect these assumptions (INE 2012b).

- The mortality scenario is defined in terms of the age- and sex-specific probabilities of death. It is assumed that life expectancy at birth will increase from 80 years in 2011 to 87 years in 2051 for men, and from 83 years to 91 years for women over the same time period. Corresponding to these assumptions, INE presents a series of age- and sex-specific probabilities of death, $\mathbf{q}(t)$ for $t = 1, \ldots, 40$ (INE 2012b).

- Migration assumptions are expressed in terms of age- and sex-specific immigration numbers and emigration rates. INE assumes that the migratory balance of Spain, which was negative by 50.000 persons in 2011, will recover during the projection period. In the last ten projection years, the number of persons who move to Spain is assumed to exceed emigration numbers by around 438.000 persons. Emigration rates are held constant over the entire projection interval.[3] Because of the assumptions of INE, we incorporated emigration into the matrix $\mathbf{U}$, treating emigration and mortality as two competing risks for leaving the population (INE 2012b).

In a press note on the population projections of 2012, INE emphasizes two key findings: First, the population of Spain is expected to decline from 46.2 million persons in 2012 to 41.5 million residents in 2052. Second, the population is expected to age. INE estimates that 37 percent of the population will be aged 64 or older in 2052, raising the overall dependency ratio, defined as the quotient between the population under 16 and over 64 years of age and the population aged 16 to 64, from 0.504 (in 2012) to 0.995 (in 2052). These projection results form the basis of governmental planning (INE 2012a). Analysing their sensitivity and elasticity to changes in the underlying assumptions is therefore not only relevant for the demographic research community, but also for policy makers in Spain.

[3] This seems strange to us, but is clear in the data provided by INE.

5 Sensitivity and elasticity of the population projection of Spain

The sensitivity and elasticity of the projection results can be evaluated by focusing on the population of Spain as a whole, or by analyzing the male and female population separately. Here, we use examples from both perspectives. In constructing the transition matrices $\mathbf{U}(t)$ we combined mortality and emigration as independent ways of leaving the population.[4] Let P_i be the element in the $(i+1, i)$ entry of $\mathbf{U}$; then we write

$$P_i = (1 - q_i)(1 - r_i) \tag{52}$$

where q_i is the probability of death and r_i the probability of emigrating.

5.1 Sensitivity of the total population size

Figure 1 shows the sensitivity of the total population size at terminal time T = 40 to changes in the vital rates applied in every projection year. The x-axis of the graphs shows the ages at which we perturb the vital rates; the y-axis shows the size of the effect. Figure 1 suggests that perturbations in vital rates tend to have the largest effect on the final population size if they occur at young adult ages, particularly around age 30.

Perturbations in mortality and emigration rates, combined into one set of rates P_i, have a w-shaped effect on the total population size, with effects being largest around age 30 and to a lesser extent around age 50. Increasing rates at these ages by one unit during the projection period reduces the final population size by between 1.8×10^7 and 2×10^7 units. Perturbations at other ages, especially above age 65, have a smaller effect on the final population size.

Perturbations in immigration also have the strongest effect on the final population size if they occur at young adult ages. At age 30, increasing immigration numbers by one unit, i.e. by one male and one female immigrant per projection year, increases the final population size by around 110 persons. This includes the additional immigrants themselves and their offspring. Above age 30, the effect of perturbations in immigration numbers decreases, first until age 40 where the effect

[4] The effect of perturbations in immigration and emigration assumptions cannot be tested jointly, since INE defines emigration assumptions as probabilities and immigration assumptions in terms of numbers. We chose to combine mortality and emigration data into one set of rates indicating processes of leaving the population. Other statistical offices commonly express emigration assumptions in the form of numbers. In this case, emigration numbers can be incorporated in the migration vector.

of perturbations briefly levels off, and then more rapidly above age 50. Note that sensitivities to changes in immigration are many orders of magnitude smaller than those to changes in the other vital rates. This is because immigration is measured in numbers, while mortality/emigration and fertility are per capita rates.

The sensitivity of total population size to perturbations in fertility rates shows a somewhat different age pattern. The effect of perturbations increases with age and is strongest at age 49. At this age, an increase in fertility rates by one unit across all projection years increases the final population size by around 10×10^6 units.

Overall, Figure 1 suggests that the population size in the final projection year is most sensitive to perturbations occurring at young adult ages, particularly in the case of mortality and migration. Numerically strong cohorts pass through age groups 30 to 40 at the beginning of the projection period, so that any perturbations in the vital rates concern large population numbers. The effects of perturbations also accumulate during the projection period, when population members move to older age groups. While Figure 1 allows comparisons of perturbation effects across ages, comparisons between vital rates are difficult, given that immigration assumptions are defined in terms of numbers and fertility and mortality/emigration assumptions as rates. In order to compare the effect of perturbations across vital rates, we calculate elasticities.

5.2 Elasticity of male and female population sizes

Figure 2 shows the elasticity of the Spanish population at T=40 to perturbations in mortality, fertility and migration, applied in every projection year. Here, we distinguish between the male and female population. Elasticity calculations also allow us to look at the effect of perturbations in mortality and emigration separately. Ages on the x-axis again represent the ages at which perturbations occur.

The elasticity patterns show similarities to the sensitivity results: The elasticity of male and female populations to perturbations in vital rates is strongest around ages 25 to 35. This is the case for immigration numbers, where the effects of perturbations are highest at age 28. The separate analysis for emigration rates shows that perturbations also have the strongest influence around age 30. A one per cent change in female emigration rates at this age across all projection years, for instance, reduces the final population size by 0.01 per cent. The size of effects is stronger for the male than for the female population. This is because the male population reacts to perturbations of both male and female immigration numbers and emigration rates. If the female population increases or is reduced due to perturbations in immigration or emigration, this changes the number of male offspring. The female population, by contrast, is not directly affected by perturbations in male migration in our model. The elasticity of the final male and female population sizes to perturbations in fertility reaches its highest level around age 35. The elasticity results thus confirm that projection parameters at ages 25 to 35 have to be defined with particular care if the projection outcome of interest is the final population size.

Only elasticity to mortality follows a different pattern: The effect of perturbations increases with age and is highest at 85 years for males and at around 90 years for females. One reason for the comparatively large effect of perturbations at these ages is that mortality rates are high, so that any proportional changes will have the large effects. Overall, however, it is remarkable that the proportional effect of perturbations in mortality rates on the total male and female population sizes in the final projection year is substantially smaller than the effect of perturbations in any of the other vital rates.

5.3 Elasticity of the school-age population (6 to 16 years)

Elasticities to perturbations in vital rates can not only be calculated for male, female or total population sizes, but also for subgroups of the population. Here, we calculate the elasticity of the school-age population groups in Spain (6 to 16 years, male and female persons combined) to perturbations. Again, we focus on the size of this population group at T= 40 and assume that perturbations have occurred throughout the projection period.

Figure 3 shows that perturbations in mortality rates have almost no influence on the number of school-age children in the final projection year - mortality rates are very low at ages 6 to 16 and any perturbations therefore do not matter for the development of this population group. Perturbations in immigration and emigration directly influence the size of the school age population if they occur at young ages (particularly ages 1 to 10 years). A one per cent increase in immigration numbers at age 5, for instance, would increase the number of school-age children in the final projection year by almost 0.02 percent. Perturbations in migration at ages 20 to 35 influence the school-age population through fertility. A change in the number of women in these age groups through migration influences the number of newborn children in Spain who with a delay of 6 years reach school age. Fertility has by far the largest effect on the school-age population: If the fertility rate was one per cent higher than assumed by INE during the projection period at age 34 alone, the school-age population in the final projection would be 0.08 per cent larger. Fertility assumptions must therefore be of particular concern for policy makers interested in the future development of this population group.

5.4 Elasticity of population with dementia

Sensitivities and elasticities to perturbations in vital rates can also be calculated for the Spanish population weighted by a set of prevalences. Here, we calculate the elasticity of the number of persons with dementia in the final projection year to perturbations in the vital rates and prevalences. Figure 4 shows the prevalence of dementia by age among the Spanish population in 2012. Prevalence rates increase strongly above age 70, with prevalence rates of women reaching higher levels than those of men. We have projected the number of persons with dementia in Spain by keeping these rates constant. Figure 4 shows the elasticity of the projected population with dementia in 2052 (male and female cases combined) to perturbations.

The number of persons with dementia reacts most strongly to perturbations in the prevalences. A one percent increase at any age between 85 and 90 years across projection years, for instance, would increase the number of dementia cases in the last projection year by between 0.05 and 0.06 per cent. Perturbations in the vital rates would have a comparatively smaller effect. Mortality and migration perturbations under age 30 do not affect the number of dementia cases in 2052 at all, since persons in these age groups do not reach ages during the projection period at which dementia becomes prevalent. For the same reason, perturbations in fertility do not influence the number of dementia cases. Above age 30, the effect of perturbations in mortality, emigration and immigration increases and reaches its highest level at ares 55 (emigration) and 65 (immigration). Perturbations of mortality show the largest impact between ages 85 and 90, when prevalence rates in dementia reach high levels. Overall, however, developments in the prevalence of dementia appear to be more decisive for the future number of dementia cases than trends in the vital rates.

5.5 Elasticity of dependency and support ratios

One of the findings that INE highlights in their press note is that the overall dependency ratio in Spain (defining persons under age 16 and over age 64 as dependent) will double during the projection period. In 2052, the dependent population in Spain is expected to be as large as the population of working age. Again, we calculate how sensitive this result is to perturbations in the vital rates. Figure 5 shows the elasticity of the dependency ratio in the final projection year to perturbations in the vital rates during the projection period.

The dependency ratio reacts to perturbations in vital rates across all ages, but the size and direction of effects differ: Perturbations in immigration and emigration between ages 20 and 30 have the strongest influence. Immigration numbers and emigration rates are particularly high among these age groups, so that proportional changes have a strong impact. In addition, cohorts who pass through these age groups particularly at the beginning of the projection period spend a large number of years in the working age population and barely contribute to to the size of the population classified as 'dependent'. Perturbations in mortality rates have a comparatively smaller effect. The elasticity of the dependency ratio increases above age 40 and reaches the highest point at age 85, when a one percent increase in the mortality rate decreases the total dependency ratio in the final projection year by 0.007 per cent. Perturbations in fertility rates have the proportionally smallest effect on the overall dependency ratio in the last projection year. This is because during the 40-year projection period, newborn cohorts contribute both to the size of age groups defined as dependent and to the working age population. Both effects largely cancel each other out.

The dependency ratio as defined by INE is a simplified construct to measure economic dependency. It disregards that population members above age 65 may continue to be productive and that not all population members aged 16 to 64 are part of the labour force. A more nuanced perspective is possible by using age-specific income and consumption data for Spain which have been prepared by the National Transfer Accounts (NTA) Project.[5] Here we have calculated support ratios for Spain which draw on these data. We use per capita normalised annual consumption (public and private consumption) and labour income flow values. The data used by NTA date from the year 2000. We have used these values as weights which we apply to the age distribution of the Spanish

[5]Data and further information are available at www.ntaccounts.org.

population in every projection year. We then obtain support ratios by calculating the ratio of income to consumption. Again, elasticities can be calculated for this more nuanced measure.

Figure 5 shows how perturbations in the vital rates, occurring in every projection year, would influence the support ratio in 2052. Due to differences in the calculation of the support ration and the dependency ratio, perturbations have opposing effects on the two indicators - any perturbation that increases the dependency ratio would decrease the support ratio. To facilitate comparisons between the two figures, we have reversed the sign of elasticity results in the support ratio figure. Perturbations show a similar pattern across ages as in case of the overall dependency ratio, with effects of perturbations across age groups largely pointing in the same direction. Effects are however smaller. This reflects that population members across the age spectrum contribute both to consumption and income patterns. Only perturbations in fertility have a qualitatively different effect from the first dependency indicator: They increase the support ratio in the last projection year. This result reflects Spanish income and consumption data which show that consumption outweighs income until age 24. Young persons therefore remain 'net consumers' for longer than assumed by the first indicator. Perturbations in fertility during a projection period of 40 years therefore put an upward pressure on the support ratio.

6 Discussion

6.1 Sensitivity analysis and scenarios

Population projections incorporate large amounts of demographic information. The projection of Spain, with 101 ages projected across 40 years on the basis of annual rates of mortality, fertility, immigration, and emigration, contains over 16,000 pieces of information. It requires some kind of parameterization carrying enough information to specify all these.

The result of this collection of demographic information is a diverse set of outcomes: population vectors, population sizes (weighted in various ways), ratios, growth rates, etc. Changes in any of the parameters at any time will change these results. The sensitivity structure quantifies these effects.

Disciplines in which sensitivity analyses of various kinds are common (e.g., population ecology from the 1980s onwards) experience a kind of shift in perspective, in which the sensitivity of a dependent variable to changes in parameters becomes as much a part of the results as the dependent variable itself. Until you have understood the sensitivity relationships, you have not understood the model.

Statistical offices and agencies often carry out projections under multiple scenarios (low, medium, high ...). Such projections are a kind of perturbation analysis, measuring the effects of large changes imposed on many of the vital rates. But there are an infinite number of possible scenario modifications. The results of a sensitivity or elasticity analysis give a quantitative measure of the effects of perturbations of specific rates. For example, from graphs of the form of Figures 4 or 5, we know, without the need for any scenario modifications at all, that changes in the vital rates will have less effect on the number of persons with dementia than changes in the prevalence rates, or that changes in fertility scenarios will have different effects on the economic support ratio than on the total dependency ratio. In addition, not only do we know that changes in the migration scenarios at different ages will have different effects on the support ratio (that's probably pretty intuitively obvious), we can say what those differences are. Such conclusions may help decide what kind of scenario modifications are most worth looking at.

6.2 Sensitivity analysis and uncertainty

Because population projections are used for many types of social, economic and ecological planning, demographers have invested considerable attention in the last years to measure their uncertainty. A large body of literature has focused on probabilistic population projections based on past projection errors, expert opinion or stochastic models (Keilman et al. 2002).

Sensitivity analysis does not, by itself, provide information on the uncertainty of a projection (it is a prospective, not a retrospective, perturbation analysis, in the terminology of Caswell (2000)). Knowing that an outcome is more or less sensitive to some parameter does not tell whether the outcome is more or less certain. Much depends on the precision with which the parameter is estimated.

Sensitivity analysis however provides a powerful way to translate uncertainty in parameter estimates into uncertainty in projection outcomes. Suppose that $\boldsymbol{\xi}$ is a projection result (vector- or scalar-valued), and that the projection depends on some set of parameters $\boldsymbol{\theta}$. The uncertainty of the estimate of $\boldsymbol{\xi}$ can be measured by the covariance matrix

$$C(\boldsymbol{\xi}) = \left(\ \mathrm{Cov}(\xi_i, \xi_j) \ \right) \tag{53}$$

If ξ is a scalar, this is simply the variance $V(\xi)$, but if the projection result is multivariate (as it often is), the covariances are an important part of the uncertainty.

The uncertainty in the estimates of the parameter vector $\boldsymbol{\theta}$ is given by the covariance matrix $C(\boldsymbol{\theta})$. This covariance matrix might be obtained, e.g., from the Fisher information matrix provided by maximum likelihood estimation of $\boldsymbol{\theta}$.

Then, to first order, the uncertainty in $\boldsymbol{\theta}$ translates into uncertainty in $\boldsymbol{\xi}$ by

$$C(\boldsymbol{\xi}) = \frac{d\boldsymbol{\xi}}{d\boldsymbol{\theta}^{\mathsf{T}}} \, C(\boldsymbol{\theta}) \, \left(\frac{d\boldsymbol{\xi}}{d\boldsymbol{\theta}^{\mathsf{T}}} \right)^{\mathsf{T}} \tag{54}$$

If ξ is a scalar, this reduces to

$$V(\xi) = \frac{d\xi}{d\boldsymbol{\theta}^{\mathsf{T}}} \, C(\boldsymbol{\theta}) \, \left(\frac{d\xi}{d\boldsymbol{\theta}^{\mathsf{T}}} \right)^{\mathsf{T}} \tag{55}$$

and if θ is also a scalar, then

$$V(\xi) = \left(\frac{d\xi}{d\theta} \right)^2 V(\theta). \tag{56}$$

These calculations formalize the intuitive notion that uncertainty in a parameter to which an outcome is very sensitive will create a high degree of uncertainty in that outcome.

6.3 Immigration and emigration

Births, deaths, and emigration are events that happen to individuals in the population under study. They can be described by rates, estimated from the number of events and the number of individuals at risk. Those rates can be transformed to probabilities and then applied to the appropriate components of cohorts to project the population forward.

Immigration, however, is not an event to which individuals in the population are at risk, and hence it cannot be described as a rate. Thus, in equations (4), (7), and (8), immigration appears as a vector $\mathbf{b}(t)$, with units of numbers of individuals, which is added to the result of applying the per capita rates in $\mathbf{U}$ and $\mathbf{F}$.

Immigration is handled differently by the various agencies and organizations engaged in projections. The projection of Spain in Section 4 has taken the entirely sensible approach of separating emigration and immigration, including the former, along with mortality, in the matrix $\mathbf{U}$, and adding the latter to $\mathbf{b}$.

The projections prepared by Eurostat (Lanzieri 2009) make this approach slightly more subtle, noting that individuals that immigrate during (t,t+1) spend some fraction of the interval in the population, and hence subject to the mortality and fertility rates in action during that time (G. Lanzieri, personal communication). This means that a basic projection equation becomes

$$\mathbf{n}(t+1) = \mathbf{A}(t)\mathbf{n}(t) + \mathbf{B}(t)\mathbf{b}(t) \tag{57}$$

where $\mathbf{B}(t)$ is a matrix that includes mortality and fertility of immigrants during the fraction of the interval during which they are assumed to be present (usually 0.5 years). The projection (57) is easily subjected to perturbation analyses. For example, the term $d\mathbf{b}(t)/d\boldsymbol{\theta}^{\mathsf{T}}(x)$ in equation (13) would simply be replaced with

$$\mathbf{B}(t)\frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \left(\mathbf{b}^{\mathsf{T}}(t) \otimes \mathbf{I}\right)\frac{d\text{vec}\,\mathbf{B}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}.$$

Another common approach is to define $\mathbf{b}$ as *net* migration (immigration - emigration); treating both immigration and emigration as additive. This has unfortunate theoretical properties; it asserts that the number of individuals leaving the population is independent of the population at risk of leaving. In principle, in the long run this could draw a population down to impossible negative values. For the short time horizons in practical population projections, this is unlikely to be a problem.

Yet another option is to describe both immigration and emigration as rates applied to the population at risk. This conceptualizes immigration as a flow of individuals "sucked" into the population by the residents. It also has bad long-run theoretical properties: the number of immigrants goes to zero as population decreases, and increases without bounds as the population grows. An empty population would remain so.

6.4 Data requirements and applications

Goldstein and Stecklov (2002) have lamented the lack of clarity and transparency in reports of population projections. The trajectories of mortality, fertility, and immigration are seldom reported, and "even when extensive documentation is provided, it is difficult to replicate the calculations without access to proprietary computer software used by the team that prepared the projection." (Goldstein and Stecklov 2002, p. 121). We urge agencies to consider reporting their projections in the form of projection matrices. The entries of $\mathbf{U}$, $\mathbf{F}$, and $\mathbf{b}$ may require considerable effort to obtain, and sophisticated methods to estimate from data on populations, births, deaths, etc. But once the estimation process is completed, the projection matrix formulation provides a readily computable, non-proprietary method of studying the results. And the mathematical relationships extracted from those matrices are valid regardless of how the matrices themselves are obtained. Sensitivity analysis is just one of the possible uses of the matrices.

Sensitivity analyses using matrix calculus techniques require only the basic ingredients of any cohort component projection — initial age- and sex-specific population vector and the fertility, mortality, and migration parameters for every projection year. The sensitivity and elasticity analyses can be extended to multistate population projections; these developments are left for future research. In the meantime, the analyses presented here will be beneficial for demographers and government officials producing projections, because they will improve our understanding of the underlying mechanisms leading to uncertainties and allow for precise quantifications of the impact of changes in vital rates or policies on any projection output.

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A Derivations

In this section, we present the derivations of the sensitivity results in Section 3.

For more details and many more demographic examples of this approach, see Caswell (2008, 2009). For an introductory presentation of the matrix calculus methods, see Abadir and Magnus (2005).

A.1 Derivatives of $\mathbf{n}(t)$

One-sex projections. We begin with the single sex projection of equation (4). Take the differential of both sides to obtain

$$d\mathbf{n}(t+1) = \mathbf{A}(t)d\mathbf{n}(t) + [d\mathbf{A}(t)]\,\mathbf{n}(t) + d\mathbf{b}(t) \tag{A-1}$$

Applying the vec operator to both sides, using the result (Roth 1934) that $\mathrm{vec}\,\mathbf{ABC} = (\mathbf{C}^{\mathsf{T}} \otimes \mathbf{A})\,\mathrm{vec}\,\mathbf{B}$, yields

$$d\mathbf{n}(t+1) = \mathbf{A}(t)d\mathbf{n}(t) + \left(\mathbf{n}^{\mathsf{T}}(t) \otimes \mathbf{I}\right) d\mathrm{vec}\,\mathbf{A}(t) + d\mathbf{b}(t) \tag{A-2}$$

Now let $\mathbf{A}(t)$ and $\mathbf{b}(t)$ be functions of the parameter vector $\boldsymbol{\theta}(x)$. By the chain rule for matrix calculus, the derivative with respect to the $\boldsymbol{\theta}$ is then

$$\frac{d\mathbf{n}(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{A}(t)\frac{d\mathbf{n}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \left(\mathbf{n}^{\mathsf{T}}(t) \otimes \mathbf{I}\right) \frac{d\mathrm{vec}\,\mathbf{A}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \frac{d\mathbf{b}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{A-3}$$

This is a dynamic system in the derivative matrix $d\mathbf{n}(t)/d\boldsymbol{\theta}^{\mathsf{T}}(x)$. If the parameter vector affects the vital rates but not the starting population for the projection, then (A-3) is iterated from the initial condition

$$\frac{d\mathbf{n}(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{0}_{\omega \times p}. \tag{A-4}$$

Setting $\boldsymbol{\theta} = \mathbf{n}_0$ gives the sensitivity of the projection to the initial population. The last two terms in (A-3) are zero, and the remaining term is iterated from the initial condition

$$\frac{d\mathbf{n}(0)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{I}_{\omega}. \tag{A-5}$$

Two-sex projections. We apply the same approach to the two-sex projection in equations (7) and (8). For notational convenience, we temporarily suppress the time-dependence of the matrices $\mathbf{U}(t)$, $\mathbf{F}(t)$, and $\mathbf{b}(t)$. Differentiating both sides of (7) and (8) gives

$$\begin{aligned} d\mathbf{n}_f(t+1) &= (d\mathbf{U}_f)\,\mathbf{n}_f(t) + \mathbf{U}_f d\mathbf{n}_f(t) + r\,(d\mathbf{F})\,\mathbf{n}_f \\ &\quad + r\mathbf{F} d\mathbf{n}_f(t) + d\mathbf{b}_f \qquad \text{(A-6)} \\ d\mathbf{n}_m(t+1) &= \mathbf{U}_m d\mathbf{n}_m(t) + (1-r)\mathbf{F} d\mathbf{n}_f(t) + (d\mathbf{U}_m)\,\mathbf{n}_m(t) \\ &\quad + (1-r)\,(d\mathbf{F})\,\mathbf{n}_f(t) + d\mathbf{b}_m \qquad \text{(A-7)} \end{aligned}$$

Applying the vec operator gives

$$\begin{aligned} d\mathbf{n}_f(t+1) &= (\mathbf{U}_f d + r\mathbf{F})\, d\mathbf{n}_f(t) + \left(\mathbf{n}_f^{\mathsf{T}}(t) \otimes \mathbf{I}\right) d\text{vec}\,\mathbf{U}_f \\ &\quad + r\left(\mathbf{n}_f^{\mathsf{T}}(t) \otimes \mathbf{I}\right) d\text{vec}\,\mathbf{F} + d\mathbf{b}_f \qquad \text{(A-8)} \\ d\mathbf{n}_m(t+1) &= \mathbf{U}_m d\mathbf{n}_m(t) + (1-r)\mathbf{F} d\mathbf{n}_f(t) + (\mathbf{n}_m^{\mathsf{T}}(t) \otimes \mathbf{I})\, d\text{vec}\,\mathbf{U}_m \\ &\quad + (1-r)\left(\mathbf{n}_f^{\mathsf{T}}(t) \otimes \mathbf{I}\right) d\text{vec}\,\mathbf{F} + d\mathbf{b}_m \qquad \text{(A-9)} \end{aligned}$$

Notice that the male population is sensitive to changes in the parameters of the female population, because of fertility. The second and fourth terms in (A-9) provide the required links between the female and male population.

Finally, we introduce the parameter vector $\boldsymbol{\theta}$ and use the chain rule to obtain the sensitivity of the two-sex projection, first for females:

$$\begin{aligned} \frac{d\mathbf{n}_f(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} &= \Big(\mathbf{U}_F(t) + r\mathbf{F}(t)\Big) \frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + \left(\mathbf{n}_f^{\mathsf{T}}(t) \otimes \mathbf{I}\right) \left(\frac{d\text{vec}\,\mathbf{U}_F(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + r \frac{d\text{vec}\,\mathbf{F}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \right) \\ &\quad + \frac{d\mathbf{b}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \qquad \text{(A-10)} \end{aligned}$$

and then for males:

$$\frac{d\mathbf{n}_m(t+1)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} = \mathbf{U}_m(t) \frac{d\mathbf{n}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + (1-r)\mathbf{F}(t) \frac{d\mathbf{n}_f(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} + (\mathbf{n}_m^{\mathsf{T}}(t) \otimes \mathbf{I}) \frac{d\text{vec}\,\mathbf{U}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}$$

$$+(1-r)\left(\mathbf{n}_f^{\mathsf{T}}(t)\otimes\mathbf{I}\right)\frac{d\text{vec}\,\mathbf{F}(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)}+\frac{d\mathbf{b}_m(t)}{d\boldsymbol{\theta}^{\mathsf{T}}(x)} \tag{A-11}$$

A.2 Derivatives of projection matrices

We turn now to the derivatives of the projection matrices $\mathbf{U}$ and $\mathbf{F}$, and the immigration vector $\mathbf{b}$, given in Section 3.4. We consider the derivatives with respect to mortality, fertility, and immigration.

Mortality. Write the matrix $\mathbf{U}$ as

$$\mathbf{U}=\mathbf{Z}\circ\left(\mathbf{1}\mathbf{p}^{\mathsf{T}}\right). \tag{A-12}$$

Differentiating gives

$$d\mathbf{U}=\mathbf{Z}\circ\left(\mathbf{1}d\mathbf{p}^{\mathsf{T}}\right). \tag{A-13}$$

Apply the vec operator

$$\begin{aligned} d\text{vec}\,\mathbf{U} &= \mathcal{D}(\text{vec}\,\mathbf{Z})\text{vec}\,\left(\mathbf{1}d\mathbf{p}^{\mathsf{T}}\right) && \text{(A-14)}\\ &= \mathcal{D}(\text{vec}\,\mathbf{Z})\left(\mathbf{I}\otimes\mathbf{1}\right)d\mathbf{p} && \text{(A-15)}\\ & && \text{(A-16)}\end{aligned}$$

The differential of $\mathbf{p}$ is

$$d\mathbf{p}=-\mathcal{D}(\mathbf{p})d\boldsymbol{\mu}. \tag{A-17}$$

Substituting (A-17) into (A-15) gives the result (24).

Fertility. The matrix $\mathbf{F}$ can be written

$$\mathbf{F}=\mathbf{e}_1\mathbf{f}^{\mathsf{T}}. \tag{A-18}$$

Differentiating gives

$$d\mathbf{F}=\mathbf{e}_1 d\mathbf{f}^{\mathsf{T}} \tag{A-19}$$

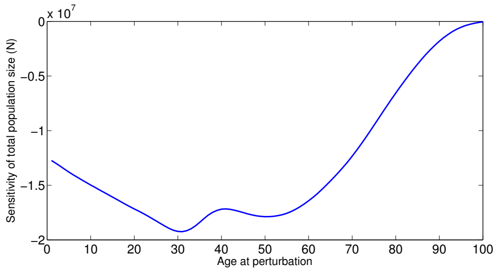
Applying the vec operator gives

$$d\mathrm{vec}\,\mathbf{F} = (\mathbf{I} \otimes \mathbf{e}_1)\, d\mathbf{f}. \tag{A-20}$$

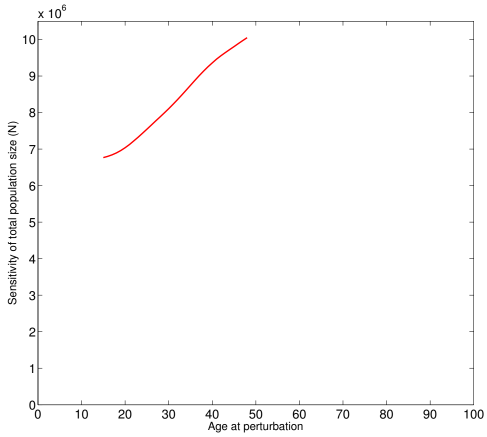
Using the delta function gives the result (25).

Immigration. The derivative of the immigration vector to itself is the identity matrix, by definition.

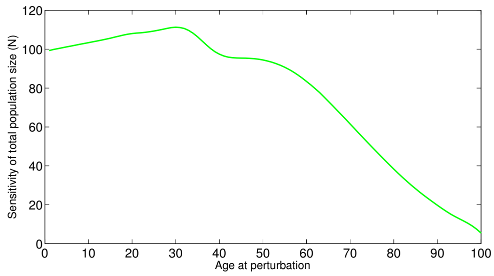
B Figures



(a) Sensitivity to mortality and emigration

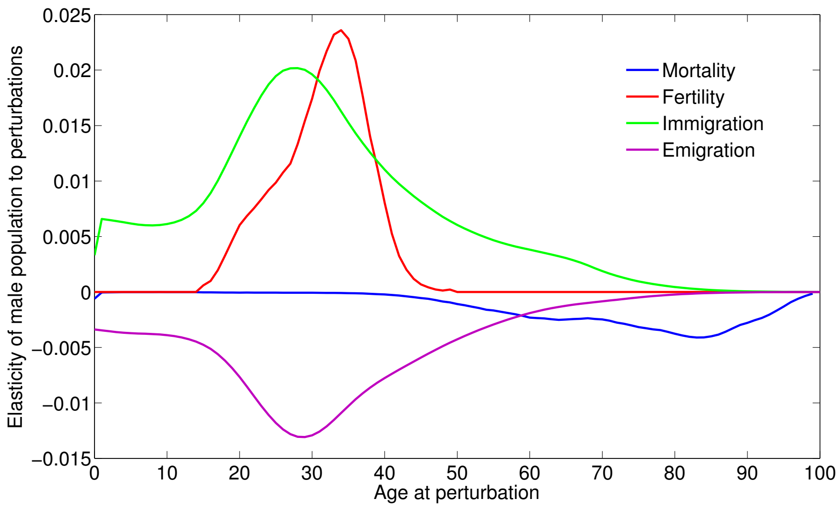


(b) Sensitivity to fertility

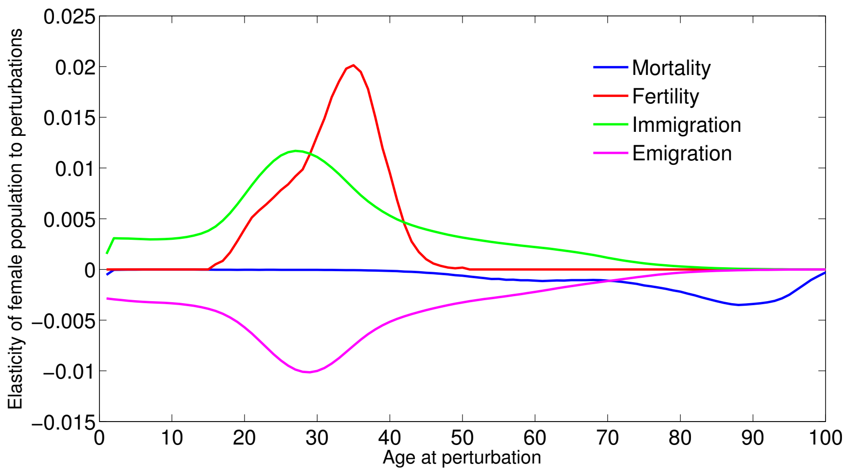


(c) Sensitivity to immigration

Figure 1: The sensitivity of $\mathbf{N}(T)$, where $T = 40$, to a change in age-specific vital rates, applied in every year from $t = 0$ to $t = T$. (a) mortality and emigration, (b) fertility, (c) immigration. Based on INE (2012) projections for Spain from 2012 to 2052.



(a) Male population



(b) Female population

Figure 2: The elasticity of male and female population size $N(t)$, where $T = 40$, to changes in age-specific vital rates, applied in every year from $t = 0$ to $t = T$. (a) male population, (b) female population. Based on INE (2012) projections for Spain from 2012 to 2052.

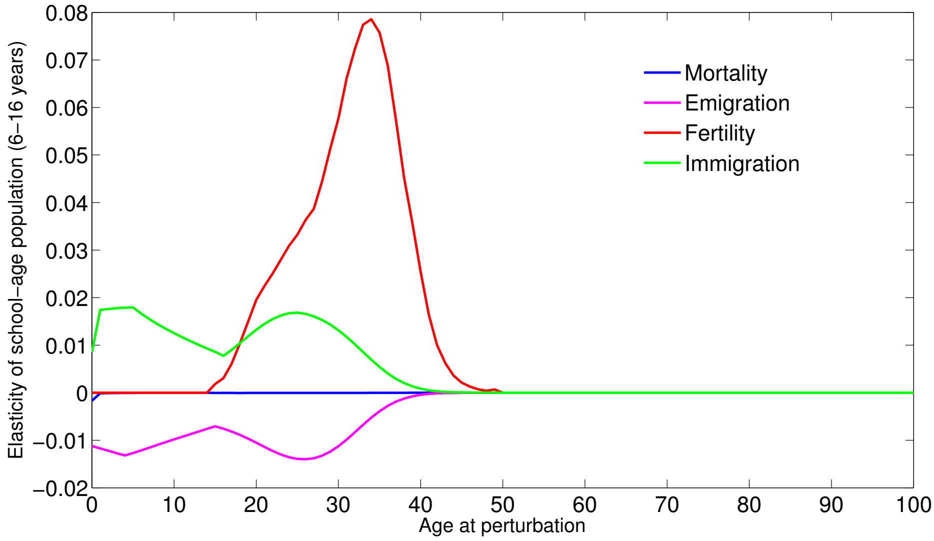
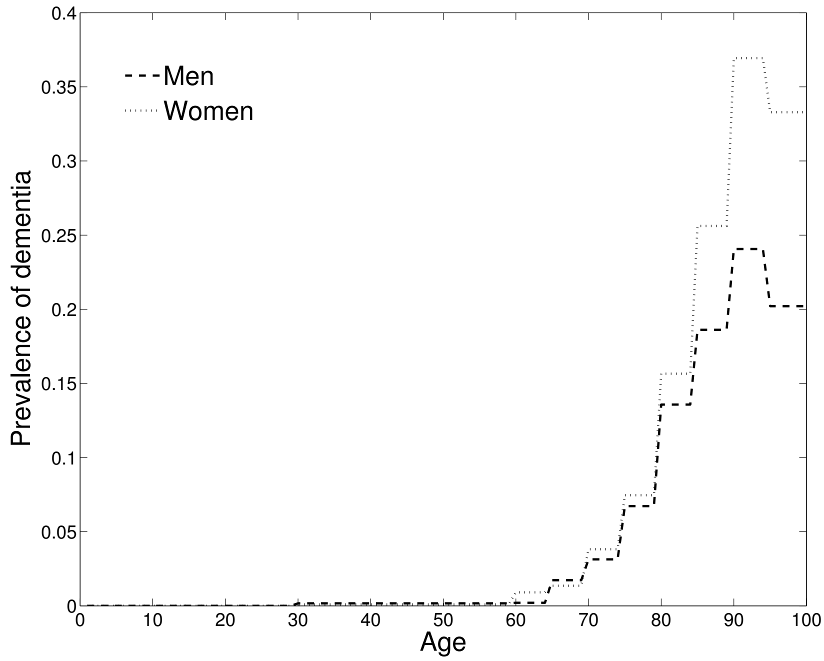
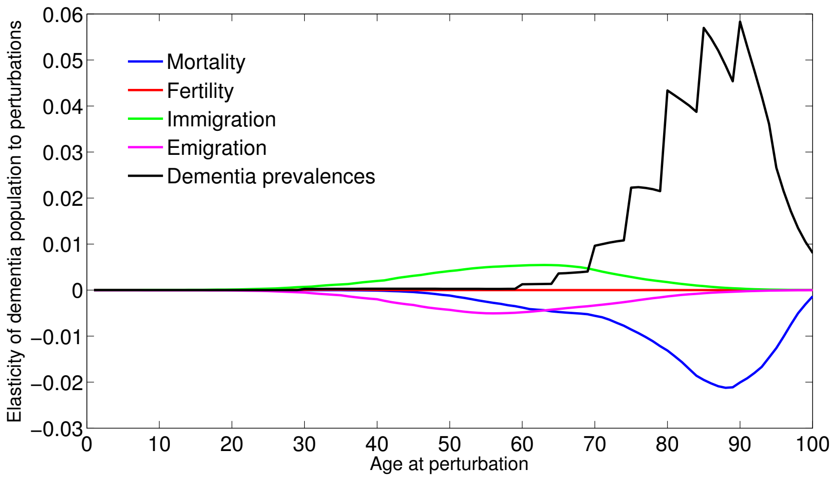


Figure 3: The elasticity of the school-age population size (6 to 16 years) at $N(t)$, where $T = 40$, to changes in age-specific vital rates, applied in every year from $t = 0$ to $t = T$. Based on INE (2012) projections for Spain from 2012 to 2052.

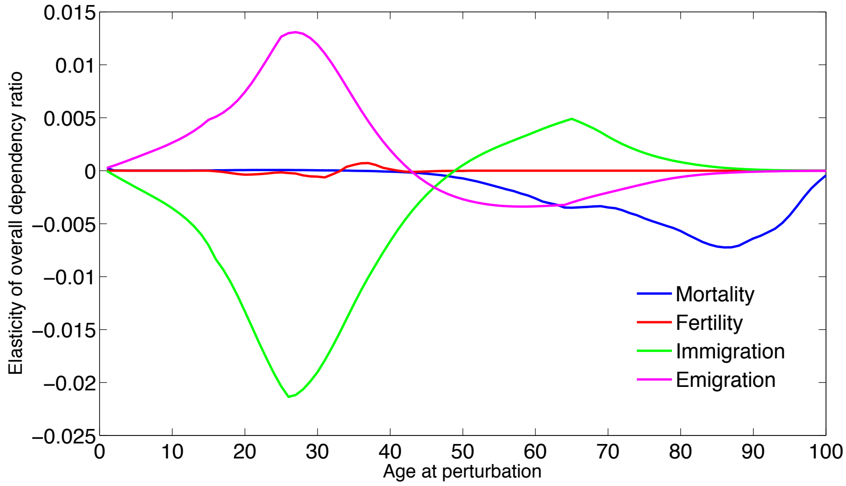


(a) Prevalence of dementia in Spain, 2012

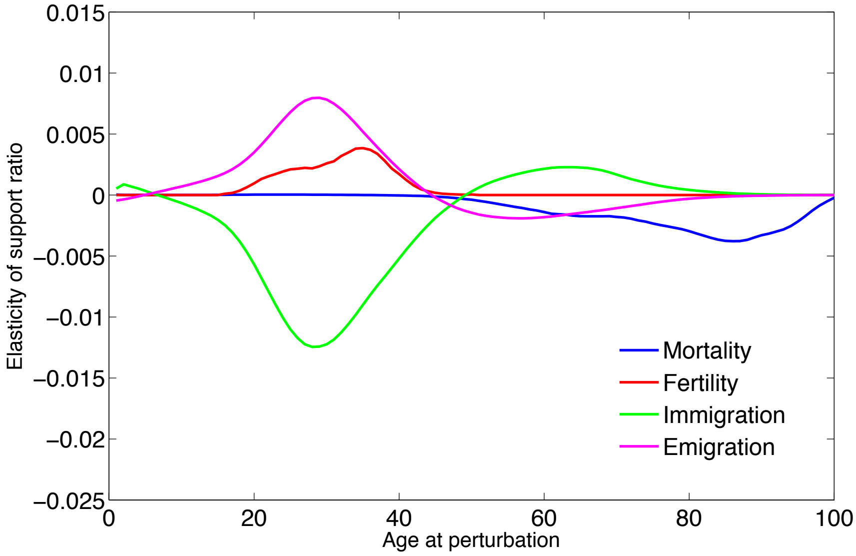


(b) Elasticity of the population with dementia

Figure 4: The elasticity of the population with dementia to changes in age-specific vital rates and prevalences at $T = 40$. The perturbations are applied in $t = 0$ to $t = T$. (a) Age- and sex-specific prevalence of dementia in Spain, (b) elasticity of population with dementia. Based on INE (2012) projections for Spain from 2012 to 2052. Data on dementia obtained from Alzheimer Europe (2014).



(a) Total dependency ratio



(b) Support ratio

Figure 5: The elasticity of the total dependency ratio and support ratio to changes in age-specific mortality, fertility, and migration at $T = 40$. The perturbations are applied in $t = 0$ to $t = T$. (a) total dependency ratio (dependent ages: below age 16 and above age 64), (b) support ratio. Based on INE (2012) projections for Spain from 2012 to 2052. Age-specific consumption and labour income data obtained from the National Transfer Accounts Project: http://ntaccounts.org