Evaluating Mortality Forecasts Using Taylor's Power Law

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Abstract

In 1961, Taylor established a power law that describes a linear relationship of log variance to log mean for population density; Taylor's law has been verified for many species in ecology, and recently for Norway's population density in human demography. In this paper, we show that Taylor's law also appears to be a regular pattern in human mortality data and how it could be used to compare and evaluate mortality forecasts. To do this, we forecast mortality for twelve countries of the Human Mortality Database from 1991 to 2009, given data from 1965 to 1990, applying different approaches like the canonical Lee-Carter model, some of its extensions and our model. The results of these retrospective forecasts suggest that only recently developed approaches can capture dynamic changes of mortality appropriately and that they can, therefore, substantially reduce forecast errors in comparison to previous approaches.

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1 Introduction

Mortality forecasts are typically evaluated with conventional error measures, which quantify how strongly forecasts deviate from observed data; those measures can be differentiated in terms of (1) whether they use relative differences like the (percentage) forecast error, (2) whether they use absolute (non-negative) values like the (absolute) forecast error and (3) whether they average deviations across age or time as the (mean) forecast error. Overviews of such error measures can be found in, e. g., Hyndman and Koehler (2006), Smith et al. (2001) or Keilman and Pham (2004). These conventional error measures can be used only some years *after* the mortality forecasts have been generated, when the forecasted values can be compared to observed data. To overcome this drawback, we suggest a novel criterion that can be used to evaluate mortality forecasts *immediately* after their generation; in principle, we suggest to evaluate the plausibility of mortality forecasts with the continuance of regular mortality patterns; if such a pattern appears to be apparent in the base period, it should be also detectable in the forecast years. In 1961, Taylor established a regular pattern in ecology; it describes a log-linear variance (Var) to mean (E) relationship for the density of sets of populations (P):

$$\log Var(P) \quad = \quad \log a + b \log E(P) \tag{1}$$

with the intercept $\log a$ and the slope b, which is sometimes, but controversially, interpreted as a species-specific *index of aggregation*.[1] Cohen et al. (2013) recently verified Taylor's power law for Norway's population — (to our knowledge) its first application in human demography. In this paper, we show that Taylor's power law also appears to apply to human mortality and we use this regular pattern to compare and to evaluate mortality forecasts for 12 countries of the Human Mortality Database (2013). To do this, we forecast female death rates from 1991 to 2009, given data from 1965 to 1990; in comparison with our model (Bohk and Rau, 2014), we use the canonical model of Lee and Carter (1992), its extensions proposed by Renshaw and Haberman (2003; 2006) as well as its coherent variant proposed by Li and Lee (2005). We select these approaches, because they differ in their capability to capture dynamic changes in mortality trends; while the original Lee-Carter model and its extensions of

[1] Kendal (2004) gives a detailed overview of the history of Taylor's law, which has been verified for hundreds of species.

Renshaw and Haberman model rather constant mortality changes between ages over time, our model and the coherent Lee-Carter model can capture mortality changes across age and time.[2]

2 Data, Methods and Results

Observed mortality Figure 1 depicts observed (black) female life expectancy at birth from 1965 to 2009 for twelve countries of the Human Mortality Database (2013), namely Denmark, France, West Germany, East Germany, Hungary, Italy, Japan, Poland, Russia, Sweden, United Kingdom and USA. For instance, Japanese female life expectancy is 86.4 years in 2009, almost 12 years longer than that of Russian women with 74.7 years at the same time. Although female life expectancy differs considerably among these twelve countries, Taylor's law can be found (across ages 0 to 100) for death rates and their rates of change in the base period from 1965 to 1990 as well as in the forecast years from 1991 to 2009 (Figures 2 and 3). Our first mortality indicator are death rates $m(x,t)$ at age x in year t; our second mortality indicator are annual rates of mortality improvement $\rho_{x,t}$, which we approximate with:

$$\rho_{x,t} = -\ln\left(\frac{m_{x,t+1}}{m_{x,t}}\right) \tag{2}$$

We then take the means (E) and variances (Var) for a given age x in a given country c across all years $t = 1, \ldots, N$.

In Figures 2 and 3, each $+$ represents the variance and the mean over all years of a death rate or a rate of mortality improvement on the logarithmic scale (with base 10) at a certain age; we depict the ages 0 to 100 with a color gradient ranging from blue for young ages to purple for old ages in the base period, and from gray for young ages to black for old ages in the forecast years.[3] We expect this remarkably regular pattern to appear only in plausible mortality forecasts.

[2]There are also other recently developed mortality forecasting approaches like the model of Li et al. (2013) or of Haberman and Renshaw (2012) that capture dynamic mortality changes across age and time. We did not include those models since we are not aware of any publicly available software to implement those methods.

[3]Further regression analyses suggest that the linear relationships are significant and that the slopes do differ among some countries (like Japan and Russia).

Forecasted mortality Figure 1 depicts forecasted life expectancy from 1991 to 2009 of the original Lee-Carter model (orange), of its variants *h0* (green) and *h2* (navy) of Renshaw and Haberman, of its coherent variant of Li and Lee (turquois) and of our model (magenta). We used the freely available package *ilc* as well as the web-based platform *lcfit* to conduct the mortality forecasts. It is clear that the approaches forecast further gains in female life expectancy in all countries between 1991 and 2009 and that the coherent model as well as our model often capture observed mortality developments more accurately than the other approaches, especially in Denmark, East Germany, Hungary and Poland. The coherent Lee-Carter model can improve its estimation by jointly forecasting mortality trends of multiple populations — a procedure that is similar to our model, which optionally allows to complement the mortality trend in a country of interest with those of selected reference countries. Such a methodological feature can be worthwhile if the long-term mortality trend is likely to change in the forecast years and if the pure extrapolation of the past trend would be, therefore, insufficient. Since we expected long-term mortality trend changes for five countries, we jointly forecasted mortality trends for Denmark (with Sweden), for East Germany (with West Germany, France and Japan), for Hungary (with West Germany), for Poland (with West Germany) and for Russia (with Belarus), and thereby enabled the coherent Lee-Carter model and our model to effectively exploit their methodological advantages.

Besides comparing observed and forecasted life expectancy, we can also test if the log-linear variance to mean relationship of the base period persists in the mortality forecasts. Figures 4 and 5 depict the observed and the forecasted variance to mean relationships of female death rates and their rates of improvement, respectively, for the different approaches. We depict observed data (ages 0 to 100) in black and forecasted data of our (Bohk and Rau, 2014) model in magenta, of the original Lee-Carter model in orange, of its coherent variant[4] in turquois and of its *h0* and *h2* extensions of Renshaw and Haberman in green and blue, respectively. It is striking that all approaches seem to develop almost linear variance to mean relationships for death rates, though our model often appears to mirror the observed pattern with smaller deviances; moreover, it is astonishing that the relationships of the models *h0* and *h2* seem to be perfectly linear in contrast to the other approaches.

Looking at the forecasted variance to mean relationships of the rates of mortality im-

[4]Please note that the coherent Lee-Carter model provides estimates only for five-year age-groups.

provement in Figure 5 can shed some light on these findings; only the coherent Lee-Carter model and our model develop variance to mean relationships that are similar to the observed ones, whereas the original Lee-Carter model develops a rather horziontal line and the extensions of Renshaw and Haberman develop rather vertical lines with almost zero variances.

From a methodological perspective, this means that the original Lee-Carter model does assume *different* mean changes of mortality for each age, whereas the extensions of Renshaw and Haberman assume rather the same mean changes of mortality for all ages. However, both models imply that these mortality changes are rather time-invariant. These methodological restrictions appear to be the reason why these models often have problems capturing changes in long-term trends and generate, therefore, larger forecast errors than the other two approaches — an effect that is particularly apparent in the mortality forecasts for Denmark, East Germany, Hungary and Poland.

3 Concluding Remarks

Our analyses suggest that (1) Taylor's law applies to human mortality data and that (2) this remarkably regular pattern can be used as a first indicator to compare and to evaluate the plausibility of forecasts immediately after their generation.

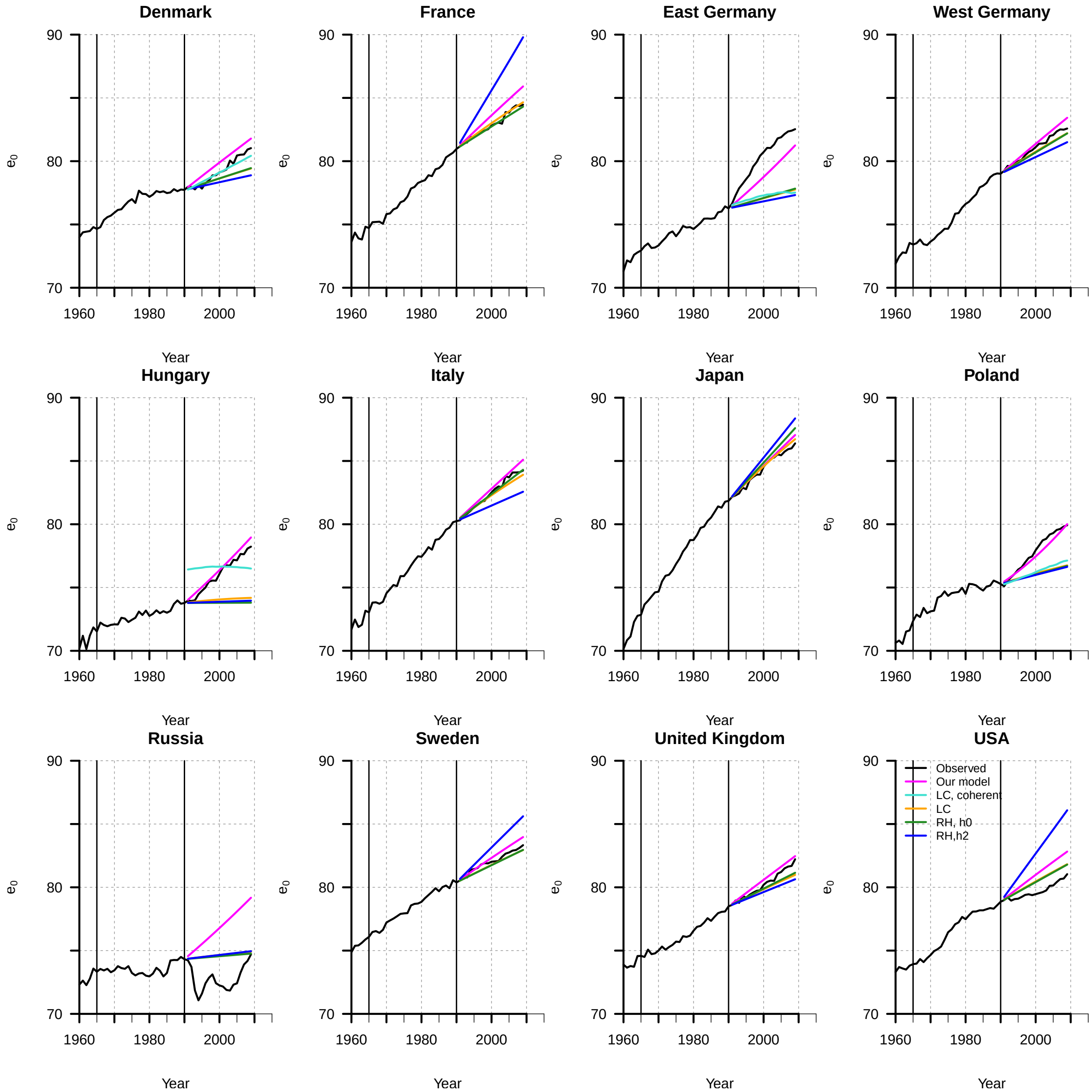

Figure 1. Observed (black) and forecasted female life expectancy at birth e_0 in twelve countries of the Human Mortality Database (2013) from 1965 to 2009. Mortality forecasts from 1991 to 2009 are depicted in orange for the original Lee-Carter model, in green and navy for its variants *h0* and *h2* of Renshaw and Haberman, in turquois for its coherent variant of Li and Lee and in magenta for our model (Bohk and Rau, 2014).

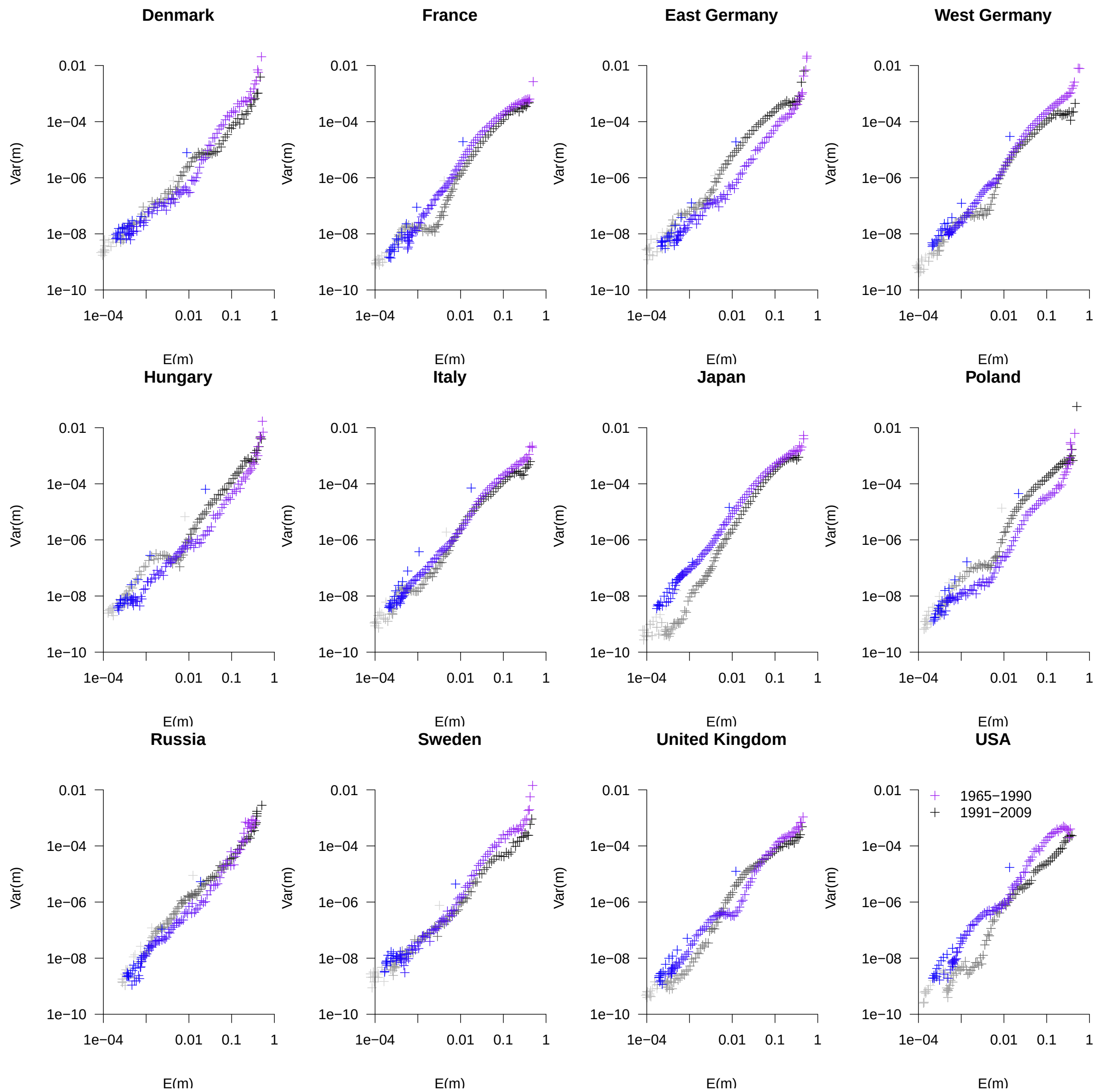

Figure 2. Log-linear variance to mean relationship for female death rates in twelve countries of the Human Mortality Database (2013) in the base period from 1965 to 1990 as well as for the forecast years from 1991 to 2009; we depict the ages 0 to 100 with the color gradient ranging from blue to purple for the base period and from gray to black for the forecast years.

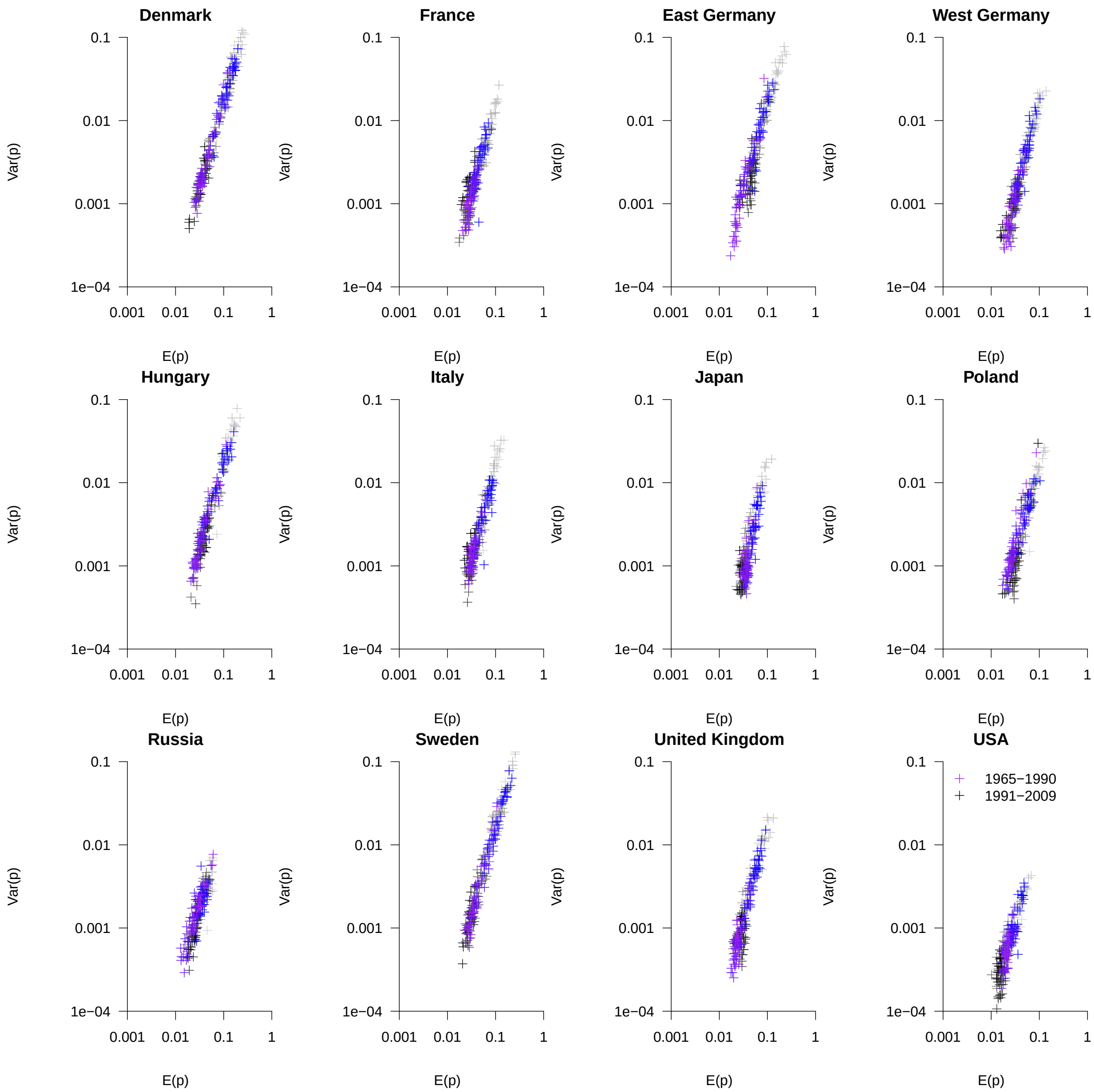

Figure 3. Log-linear variance to mean relationship for female rates of mortality improvement in twelve countries of the Human Mortality Database (2013) in the base period from 1965 to 1990 as well as for the forecast years from 1991 to 2009; we depict the ages 0 to 100 with a color gradient ranging from blue to purple for the base period and from gray to black for the forecast years.

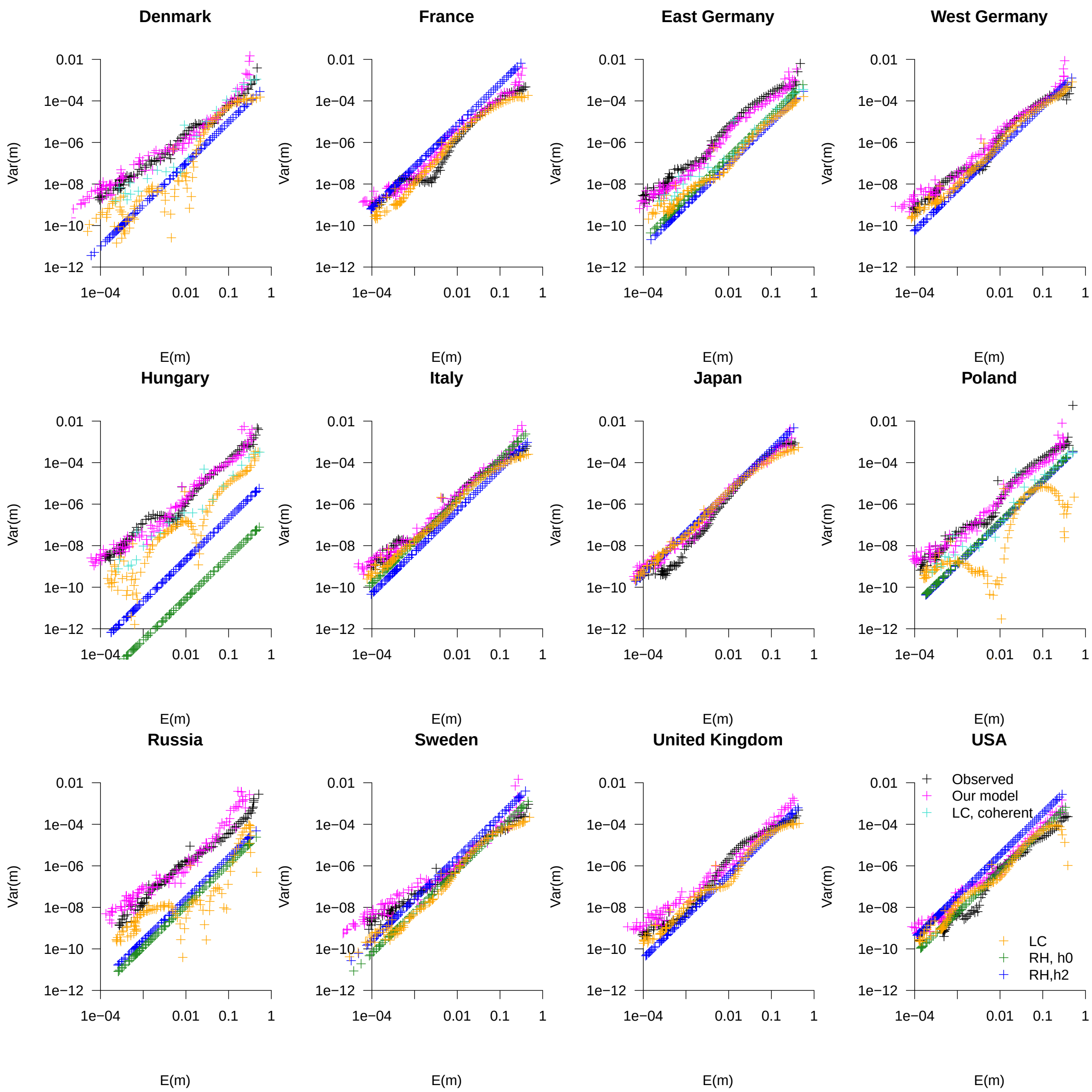

Figure 4. Variance to mean relationships of female death rates on the logarithmic scale for twelve countries of the Human Mortality Database (2013) between 1991 and 2009; we depict observed data (ages 0 to 100) in black and forecasted data of our model (Bohk and Rau, 2014) in magenta, of the original Lee-Carter model in orange, of its coherent variant in turquois and of its *h0* and *h2* extensions of Renshaw and Haberman in green and blue, respectively.

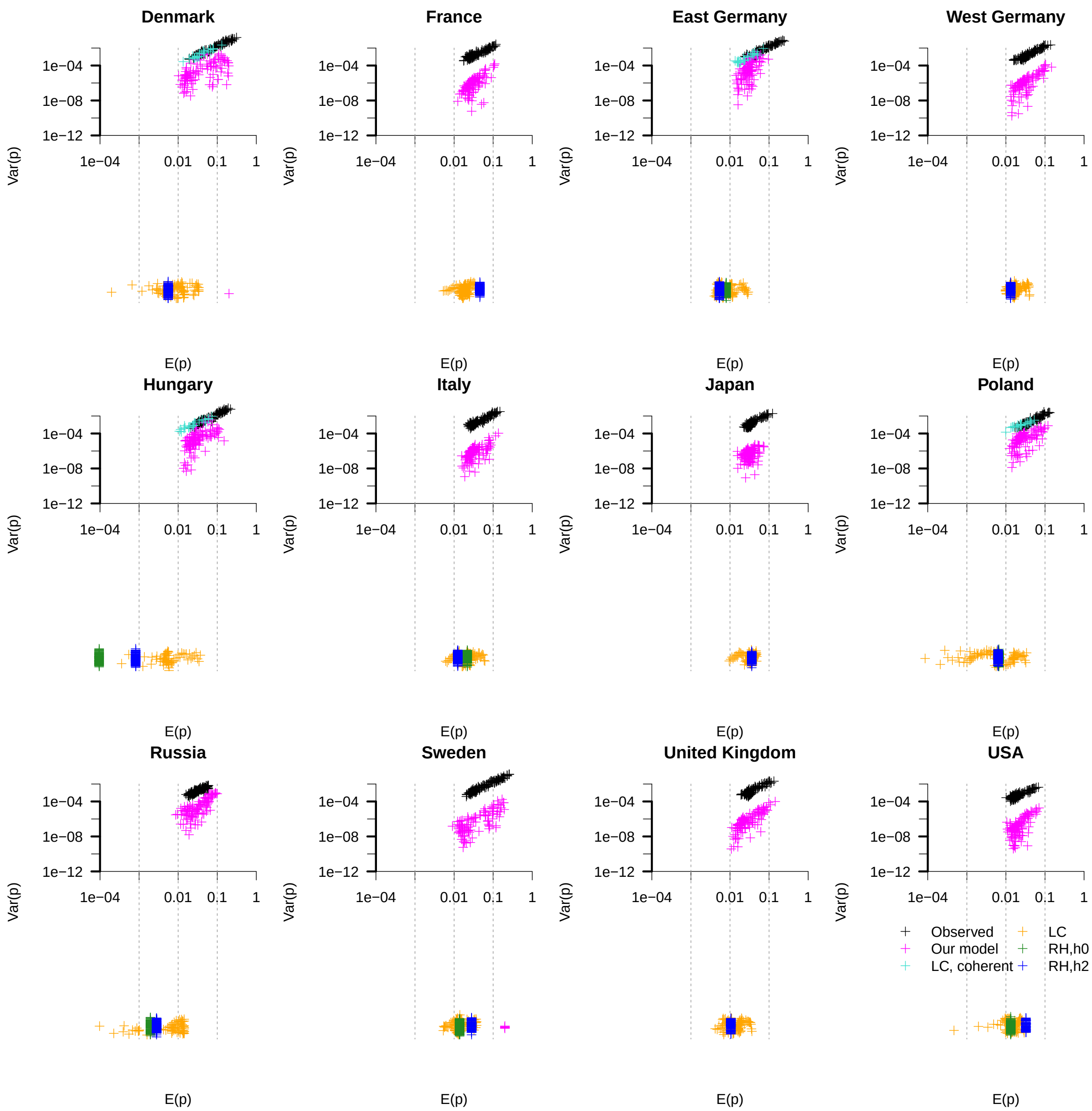

Figure 5. Variance to mean relationships of female rates of mortality improvement on the logarithmic scale for twelve countries of the Human Mortality Database (2013) between 1991 and 2009; we depict observed data (ages 0 to 100) in black and forecasted data of our model (Bohk and Rau, 2014) in magenta, of the original Lee-Carter model in orange, of its coherent variant in turquois and of its *h0* and *h2* extensions of Renshaw and Haberman in green and blue, respectively.

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